

## Modes in Layered Media

Ronald T Kessel,

School of Earth and Ocean Sciences,  
University of Victoria, PO Box 3055, Victoria BC, Canada  
Email: rkessel@leafs.seos.uvic.ca

**Introduction:** Mode models of sound propagation in layered media are now well established. Owing to its complexity, however, the theory of modes can only be pressed to completion for simple layered media consisting of just one or two homogeneous layers, while illustrating one or another subset of the many different types of modes that occur in realistic geoacoustic media. My objective is to outline a general approach for all modes that is ultimately the same as that used for analyzing modes in any linear system, and to highlight some essential rules of mode behaviour that apply in many-layered, fluid and solid media.

**Modes in linear systems:** The partial differential equations (PDE's) governing a linear system—whether acoustic, electronic, or mechanical—can often be simplified using integral transforms (such as the Fourier transform), that together with boundary conditions reduce the mathematical problem to a system of linear equations  $L$ , in selected physical variables  $\mathbf{v}$ , subject to some forcing  $\mathbf{f}$ ;

$$L\mathbf{v} = \mathbf{f}.$$

A mode in such a system is by definition a solution that persists in the absence of forcing, when  $\mathbf{f}=0$ , which occurs when the determinant  $|L|=0$ . The oscillations of the mode are given by the corresponding null space solution  $\mathbf{v}$ .

**Modes in Layered Media:** Likewise, a Fourier-Bessel transform and boundary conditions reduce the PDE's for elastic wave motion, where  $\mathbf{v}$  now holds the complex valued down (+) and up (-) going compressional (P) and shear (S), plane wave strengths at the layer interfaces, for plane waves at frequency  $\omega$  and horizontal wavenumber  $k$ ;  $L(\omega, k)$  represents the transmission and refraction of those waves through layers and interfaces; and  $\mathbf{f}$  represents the source excitation. Here again, a mode exists when  $|L(\omega, k)|=0$ . Using a computer, we must search for the roots  $\omega$  and  $k$  where this condition is satisfied. For a given  $\omega$  there is a series (possibly infinite) of modal wavenumbers  $k$ .

**Inhomogeneous Plane Waves:** The propagation of plane waves through layered

media is governed both by Snell's law and the boundary conditions applied at the interfaces between layers. In effect, Snell's law states that the horizontal wavenumber  $k$  for a plane wave must be the same in all layers. To include all possible modes and energy absorption by the media, we must consider *inhomogeneous* waves, whose horizontal and vertical wavenumbers,  $k = k_r + ik_i$  and  $\gamma = \gamma_r + i\gamma_i$ , are complex; the wave's time  $t$ , horizontal range  $r$ , and depth  $z$  dependence within a homogeneous layer going as

$$e^{i(\pm\gamma z + kr - \omega t)} = e^{i(\mp\gamma_i z - k_i r)} e^{i(\pm\gamma_r z + k_r r - \omega t)}$$

Unlike  $k$ , the vertical wavenumber  $\gamma$  changes in each layer and according to the wave type, going as  $\gamma^2(z) = \omega^2 / c^2(z) - k^2$ , where  $c(z)$  is the P- or S-wave phase speed.

**Proper Modes in the complex  $k$ -plane:**

$\gamma_i \geq 0$  for proper (physical) modes whose vibrations remain bounded as  $z \rightarrow \pm\infty$ . We can deduce where most of these proper modes must lie. If the vibrations of a mode are significantly large and span a layer (or band of similar layers) of significant thickness  $H$ , then  $\gamma_i$  must be small, otherwise the vibrations vanish because  $e^{-\gamma_i H} \rightarrow 0$ .  $k$  for such modes therefore lie close

to the line defined by  $\gamma_i = 0$  —a line that follows a portion of a hyperbola, in the first quadrant of the complex  $k$ -plane, asymptotic to the real and imaginary axes. The vibrations of proper modes whose  $k$  are far from this line must in depth be confined close to an interface. These may be Rayleigh or Scholte interface modes, or plate modes in a thin band of solid layers.

**Other Properties:** Continuing this way, we gain insight into the classification of modes as propagating or evanescent, proper or leaky, predominantly P or S, duct or interface modes, which in turn enables us to judge by inspection what modes are likely to exist in realistic media. Helpful analogies with modes in other linear systems can also be drawn.