

A finite difference scheme for wave propagation through absorbing media

Raphael Slawinski, Dept. of Geology and Geophysics, The University of Calgary
email: raphael@geo.ucalgary.ca

1. Introduction

Because the earth is not a perfectly elastic medium, oscillatory motion is damped; as a result seismic waves propagating in the earth suffer absorption. The intrinsic anelasticity of the earth is usually modeled by the linear theory of viscoelasticity, which generalizes the theory of elasticity and that of viscous fluids. The general viscoelastic equation of motion (VEM) is an integro-differential equation, and in most cases of interest must be solved numerically. We present a finite difference scheme based on the reformulation of the VEM as a higher than second order partial differential equation (PDE).

2. Elasticity and Viscoelasticity

The basic equation of motion for a solid continuum is

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2, 3, \quad (1)$$

where u_i is the particle displacement vector, σ_{ij} the stress tensor, and ρ the density; and where the summation convention is assumed. Elasticity theory follows by assuming a stress-strain relation of the form

$$\sigma_{ij} = c_{ijkl} e_{kl}, \quad (2)$$

where $e_{kl} = 1/2(\partial u_k / \partial x_l + \partial u_l / \partial x_k)$ is the strain tensor, and c_{ijkl} is the tensor of elastic constants (Aki and Richards, 1980). Viscoelasticity generalizes the above relation by allowing the stress to depend on the entire past history of the strain:

$$\sigma_{ij}(t) = \int_{-\infty}^t G_{ijkl}(t - \tau) \frac{de_{kl}(\tau)}{d\tau} d\tau \quad (3)$$

(Christensen, 1971).

For simplicity, we consider SH waves, transverse waves polarized in the horizontal plane, which for a vertically inhomogeneous medium are decoupled from the two other possible wave polarizations (P and SV). For SH waves, equation (3) becomes a scalar relation:

$$\sigma(t) = \int_{-\infty}^t R(t - \tau) \frac{du'(\tau)}{d\tau} d\tau, \quad (4)$$

where the prime denotes differentiation with respect to the spatial coordinate.

The VEM is an integro-differential equation and in general is tractable only by numerical methods. To make even a numerical solution practical, an arbitrary relaxation function $R(t)$ is approximated by a finite sum of decaying exponentials (the generalized Maxwell model) (Emmerich and Korn, 1987). By introducing auxiliary "memory" variables, the VEM may then be reformulated as a system of coupled PDEs and solved, for example, by the finite difference method.

3. Differential Formulation

Alternatively, the viscoelastic stress-strain relation (equation (4)) may be formulated using differential operators:

$$P\left(\frac{d}{dt}\right) \sigma(t) = Q\left(\frac{d}{dt}\right) u'(t), \quad (5)$$

where P and Q are polynomials of degree N in the differential operator d/dt , and where the whole of the spatial inhomogeneity is contained in the coefficients of Q . Instead of an integro-differential equation, the VEM becomes a PDE of order $N + 2$ in time:

$$\frac{\partial}{\partial x} Q\left(\frac{\partial}{\partial t}\right) \frac{\partial u}{\partial x} = \rho P\left(\frac{\partial}{\partial t}\right) \frac{\partial^2 u}{\partial t^2}. \quad (6)$$

The solution of the associated initial value problem requires the values of the function u and its time derivatives through order $N + 1$ to be specified at the initial time $t = 0$. However, the requirement that equation (5) be equivalent to (4) supplies constraints on the initial conditions so that only $u|_{t=0}$ and $\partial u / \partial t|_{t=0}$ may be freely specified. This is reasonable, since the VEM is an expression of the (second order in time) Newton's Second Law.

4. Conclusion

By replacing both space and time derivatives in equation (6) by centered finite difference expressions (while preserving the operator ordering of (6)), a stable finite difference scheme is obtained. This algorithm, including constraints on initial conditions, was then implemented in FORTRAN code. The program has been checked using a number of test calculations with known analytical or semi-analytical solutions.

While the approach via the differential formulation of the stress-strain relation does not appear to offer a computational advantage over existing approaches, it is of theoretical and conceptual interest. In particular, in this approach the time evolution of the wavefield is governed by a single PDE. By constructing a working computer code we have shown that this approach is practical for computation.

References:

- Aki, K. and Richards, P.G., 1980, Quantitative Seismology, theory and methods: W.H. Freeman and Co.
- Christensen, R.M., 1971, Theory of Viscoelasticity, An Introduction: Academic Press.
- Emmerich, H. and Korn, M., 1987, Incorporation of attenuation into time-domain computations of seismic wavefields: Geophysics, **52**, 1252-1264.