Modal Decomposition of Ocean Acoustic Fields
Using Damped Least-Squares Inversion

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Long-range propagation of acoustic pressure fields in the ocean is often well modelled as a discrete set of propagating normal modes

\[ p(r, z) = b \sum_{j=1}^{M} \phi_j(z) \phi_j(z_*) e^{ik_j r} \]

where \( p(r, z) \) is the (complex) pressure at range \( r \) and depth \( z \), \( M \) is the number of modes, \( \phi_j \) and \( k_j \) are the mode functions and wavenumbers, respectively, \( z_* \) is the source depth, and \( b \) is a complex constant. In this case, the acoustic field measured at an array of sensors can be decomposed into its modal components providing the basis for matched-mode processing techniques. The modal summation can be written as a linear matrix equation

\[ A x = p, \]

where \( A \) is the mode matrix, \( x \) represents the modal excitations, and \( p \) is the pressure measurements. For example, for a vertical array of \( N \) sensors

\[ p = [p(z_1), \ldots, p(z_N)]^T, \]

\[ A = b \begin{bmatrix} \phi_1(z_1) & \cdots & \phi_M(z_1) \\ \vdots & \ddots & \vdots \\ \phi_1(z_N) & \cdots & \phi_M(z_N) \end{bmatrix}, \]

\[ x = \begin{bmatrix} \phi_1(z_1) e^{ik_1r} / \sqrt{k_1r} \\ \cdots \\ \phi_M(z_N) e^{ik_Mr} / \sqrt{k_Mr} \end{bmatrix}^T. \]

The corresponding expressions for a horizontal array are somewhat more complicated and are range dependent.

For an overdetermined system \((N > M)\), the least-squares solution is obtained by minimizing the squared error

\[ \psi_{ls} = [Ax - p]^T[Ax - p] \]

to yield

\[ x_{ls} = [A^T A]^{-1} A^T p, \]

where \(^T\) indicates conjugate transpose. For a vertical array which densely samples the water column, the mode matrix \( A \) is approximately orthogonal, and the inversion is straightforward. However, for vertical arrays which poorly sample the water column or for horizontal arrays, \( A \) is non-orthogonal, and \( A^T A \) can be ill-conditioned, leading to instability and poor results for noisy data. This difficulty is sometimes addressed by carrying out a pseudo-inverse of \( A^T A \) using singular value decomposition and deleting the smallest singular values in an ad hoc manner.

The method of damped least-squares (DLS) provides a regularized inversion with a rigorous approach to controlling the level of misfit. In its most general form, the method is based on minimizing a functional

\[ \psi_{dls} = [G(Ax - p)]^T [G(Ax - p)] + \theta (Hx)^T (Hx). \]

The first term represents the data misfit, the second is a regularizing term, and \( \theta \) is an arbitrary parameter which controls the trade-off between the two terms. \( G \) and \( H \) represent weighting matrices for the data residuals and modal excitations, respectively. Typically, for data with uncorrelated noise, \( G \) is taken to be

\[ G = \text{diag}(1/\sigma_1, \ldots, 1/\sigma_N), \]

where \( \sigma_j \) is the standard deviation of the \( j \)th datum. \( H \) can be chosen arbitrarily to minimize different combinations of the excitations (or differences between excitations), providing flexibility in determining the character of the solution. The DLS solution is given by

\[ x_{dls} = (G A^T G A + \theta H^T H)^{-1} A^T G^T p. \]

The trade-off parameter \( \theta \) is chosen so that the (noisy) data are fit to a statistically meaningful level, e.g., to achieve a \( \chi^2 \) misfit of

\[ \chi^2 = [G(Ax_{dls} - p)]^T [G(Ax_{dls} - p)] = 2N \]

for \( N \) complex equations. Since \( \chi^2 \) is a monotonically increasing function of \( \theta \), an appropriate value of \( \theta \) can be determined efficiently using Newton's method. DLS can also be modified to determine the smallest-deviatoric solution, i.e., the solution \( x_{sd} \) which deviates minimally from an arbitrary reference vector \( x_0 \). Defining \( x = x_0 + \delta x \), the modal equations can written

\[ A \delta x = p - A x_0 = p_0. \]

Applying the DLS formalism leads to

\[ x_{sd} = x_0 + [(G A)^T G A + \theta H^T H]^{-1} A^T G^T G p_0. \]

The characteristics and potential advantages of DLS inversion for modal decomposition will be illustrated and discussed in this paper.