Modal Decomposition of Ocean Acoustic Fields Using Damped Least-Squares Inversion

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Long-range propagation of acoustic pressure fields in the ocean is often well modelled as a discrete set of propagating normal modes

$$p(r,z) = b \sum_{j=1}^{M} \phi_j(z)\, \phi_j(z_s)\, \frac{e^{ik_j r}}{\sqrt{k_j r}},$$

where $p(r,z)$ is the (complex) pressure at range r and depth z, M is the number of modes, ϕ_j and k_j are the mode functions and wavenumbers, respectively, z_s is the source depth, and b is a complex constant. In this case, the acoustic field measured at an array of sensors can be decomposed into its modal components providing the basis for matched-mode processing techniques. The modal summation can be written as a linear matrix equation

$$\mathbf{A}\,\mathbf{x} = \mathbf{p},$$

where $\mathbf{A}$ is the mode matrix, $\mathbf{x}$ represents the modal excitations, and $\mathbf{p}$ is the pressure measurements. For example, for a vertical array of N sensors

$$\mathbf{p} = [p(z_1), \ldots, p(z_N)]^T,$$

$$\mathbf{A} = b \begin{bmatrix} \phi_1(z_1) & \cdots & \phi_M(z_1) \\ \vdots & \ddots & \vdots \\ \phi_1(z_N) & \cdots & \phi_M(z_N) \end{bmatrix},$$

$$\mathbf{x} = \left[\phi_1(z_s)\frac{e^{ik_1 r}}{\sqrt{k_1 r}}, \ldots, \phi_M(z_s)\frac{e^{ik_M r}}{\sqrt{k_M r}} \right]^T.$$

The corresponding expressions for a horizontal array are somewhat more complicated and are range dependent.

For an overdetermined system $(N > M)$, the least-squares solution is obtained by minimizing the squared error

$$\psi_{\rm ls} = [\mathbf{A}\,\mathbf{x} - \mathbf{p}]^\dagger [\mathbf{A}\,\mathbf{x} - \mathbf{p}]$$

to yield

$$\mathbf{x}_{\rm ls} = [\mathbf{A}^\dagger \mathbf{A}]^{-1} \mathbf{A}^\dagger \mathbf{p},$$

where † indicates conjugate transpose. For a vertical array which densely samples the water column, the mode matrix $\mathbf{A}$ is approximately orthogonal, and the inversion is straightforward. However, for vertical arrays which poorly sample the water column or for horizontal arrays, $\mathbf{A}$ is non-orthogonal, and $\mathbf{A}^\dagger\mathbf{A}$ can be ill-conditioned, leading to instablility and poor results for noisy data. This difficulty is sometimes addressed by carrying out a pseudo-inversion of $\mathbf{A}^\dagger\mathbf{A}$ using singular value decomposition and deleting the smallest singular values in an ad hoc manner.

The method of damped least-squares (DLS) provides a regularized inversion with a rigorous approach to controlling the level of misfit. In its most general form, the method is based on minimizing a functional

$$\psi_{\rm dls} = [\mathbf{G}\,(\mathbf{A}\,\mathbf{x} - \mathbf{p})]^\dagger [\mathbf{G}\,(\mathbf{A}\,\mathbf{x} - \mathbf{p})] + \theta\,(\mathbf{H}\,\mathbf{x})^\dagger (\mathbf{H}\,\mathbf{x}).$$

The first term represents the data misfit, the second is a regularizing term, and θ is an arbitrary parameter which controls the trade-off between the two terms. $\mathbf{G}$ and $\mathbf{H}$ represent weighting matrices for the data residuals and modal excitations, respectively. Typically, for data with uncorrelated noise, $\mathbf{G}$ is taken to be

$$\mathbf{G} = \mathrm{diag}\{1/\sigma_1, \ldots, 1/\sigma_N\},$$

where σ_j is the standard deviation of the jth datum. $\mathbf{H}$ can be chosen arbitrarily to minimize different combinations of the excitations (or differences between excitations), providing flexibility in determining the character of the solution. The DLS solution is given by

$$\mathbf{x}_{\rm dls} = [(\mathbf{G}\,\mathbf{A})^\dagger \mathbf{G}\,\mathbf{A} + \theta\,\mathbf{H}^\dagger\mathbf{H}]^{-1} \mathbf{A}^\dagger \mathbf{G}^\dagger \mathbf{G}\,\mathbf{p}.$$

The trade-off parameter θ is chosen so that the (noisy) data are fit to a statistically meaningful level, e.g., to achieve a χ^2 misfit of

$$\chi^2 = [\mathbf{G}\,(\mathbf{A}\,\mathbf{x}_{\rm dls} - \mathbf{p})]^\dagger [\mathbf{G}(\mathbf{A}\,\mathbf{x}_{\rm dls} - \mathbf{p})] = 2N$$

for N complex equations. Since χ^2 is a monotonically increasing function of θ, an appropriate value of θ can be determined efficiently using Newton's method. DLS can also be modified to determine the smallest-deviatoric solution, i.e., the solution $\mathbf{x}_{\rm sd}$ which deviates minimally from an arbitrary reference vector $\mathbf{x}_{\rm O}$. Defining $\mathbf{x} = \mathbf{x}_{\rm O} + \delta\mathbf{x}$, the modal equations can written

$$\mathbf{A}\,\delta\mathbf{x} = \mathbf{p} - \mathbf{A}\,\mathbf{x}_{\rm O} \equiv \mathbf{p}_{\rm O}.$$

Applying the DLS formalism leads to

$$\mathbf{x}_{\rm sd} = \mathbf{x}_{\rm O} + [(\mathbf{G}\,\mathbf{A})^\dagger \mathbf{G}\,\mathbf{A} + \theta\,\mathbf{H}^\dagger\mathbf{H}]^{-1} \mathbf{A}^\dagger \mathbf{G}^\dagger \mathbf{G}\,\mathbf{p}_{\rm O}.$$

The characteristics and potential advantages of DLS inversion for modal decomposition will be illustrated and discussed in this paper.