Fast and accurate 3D acoustic propagation and inversion in layered media environments

James W. Wiskin, D. T. Borup, S. A. Johnson, Dept. Bio-engineering, Univ. of Utah, Salt lake City, Utah, 84112, wiskin@ee.utah.edu

An important problem in marine science is the scattering of sound by objects buried in underwater sediments. A model is desired which accounts for all orders of multiple reverberation between a scattering object and the sediment-sea interface, and is very computationally efficient. This paper will discuss a method for forward scattering from large objects in a 3D layered environment. In particular, we detail the use of a "layered Green's function" for the solution of the forward scattering problem, and can be considered as a generalization of the numerical method in [1]. There exist presently, many methods of forward scattering from an arbitrary object, such as Finite Element Methods, Finite Difference methods, Boundary Element methods, and Volume Integral equation methods. The FEM is very popular presently but has the disadvantage of requiring a large number of spatial samples per wavelength relative to our method, in order to ensure accuracy. A 2D version of this forward problem is implemented in an optimization based inversion scheme [1] developed at the Center for Inverse Problems, Imaging and Tomography (CIPIT) at the University of Utah.

We show the form of the 3-D Layered Green's function, and show how it is utilized efficiently using FFT's. After developing the 3-D Lippmann-Schwinger equation for layered media and listing its benefits and disadvantages compared with other numerical methods, several examples of forward scattering are computed using the integral equation with G_L and an independent method. The geometry relevant to the forward problem is displayed in figure 1. The grid is $nx \cdot ny \cdot nz$ pixels in size, where nx ny, nz are the number of pixels in the x, y and z directions, respectively. An inclusion is buried within the layered medium, which is completely contained within one layer. In the forward problems, the acoustic wavespeed, density, thickness, and attenuation of the layers in which the object is buried are known. We determine the total field that results when a known incident field is projected upon a scatterer with known parameter values (wavespeed etc.).

The careful development of the Green's function reveals that the BiStab-FFT method developed for the free space scattering case is applicable. The development for the Elastic equation is very similar to the acoustic case conceptually, but more difficult by virtue of its vector character, it will be addressed in a future publication. We show the layered media analogue of the Lippmann-Schwinger equation and the layered medium Green's function, we discretize the layered medium Green's function and the integral equation and discuss a fast numerical method of solution. Finally results of simulations and comparison with FDTD codes are shown and discussed.

In the case of an inhomogeneous body residing within a layered matrix, the governing equation for acoustic wave propagation in a 3D medium is:

$$\nabla^2 f(x,y,z) + k^2(x,y,z) f(x,y,z) = S_\omega(x,y,z)$$

where S_ω is the source function. The total field is broken up into scattered and incident field components: $f(x,y,z) = f^{inc}(x,y,z) + f^{sc}(x,y,z)$. The wave equation for the incident field is

$$\nabla^2 f^{inc}(x,y,z) + k_L^2(z) f^{inc}(x,y,z) = S_\omega(x,y,z)$$

Subtracting (3) from (2) the equation for the scattered field is found to be,

$$\nabla^2 f^{sc}(x,y,z) + k_L^2(z) f^{sc}(x,y,z) = -k_s^2(x,y,z) f(x,y,z)$$

where $c_O(z)$ is the speed of sound in the layered background medium, $k_L^2(z) = \omega^2 / c_L^2(z)$ is the wavenumber for the background layered medium, and $k^2(x,y,z) = \omega^2 / c^2(x,y,z)$ is the wavenumber corresponding to the spatially dependent speed of sound, $k^2(x,y,z) \equiv k_s^2(x,y,z) + k_L^2(z)$, where $k_s(x,y,z)$ is the part of the wavenumber that is due to the scattering object alone. See fig. 1. Now since $k_s^2(x,y,z) = \dfrac{\omega^2}{c_{sc}^2}\left(\dfrac{c_{sc}^2}{c^2(x,y,z)} - \dfrac{c_{sc}^2}{c_L^2(z)}\right)$, defining: $\gamma_L(x,y,z) \equiv \left(\dfrac{c_{sc}^2}{c^2(x,y,z)} - 1\right)$ where c_{sc} is the constant speed of sound of the layer which contains the inhomogeneity, gives:

$$\nabla^2 f^{sc}(x,y,z) + k_L^2(z) f^{sc}(x,y,z) = -k_{sc}^2 \gamma_L f(x,y,z)$$

The corresponding Lippmann-Schwinger Integral equation (LSIE):

$$p_{\omega\theta}^{inc}(r) = p_{\omega\theta}(r) - k_0^2 \iiint \gamma(r) p_{\omega\theta}(r) G_\omega(|r - r'|) dx'dy'dz'$$

where $c(x,y,z) = \omega / k(x,y,z)$ is the spatially varying speed of sound, $k_0^2 = \omega^2 / c_{sc}^2$ is the background wavenumber in the layer containing the scattering inhomogeneity, ω is the frequency of the interrogating field, $k(x,y,z)$ is the spatially dependent wavenumber of the body to be imaged, $p(r)$ is the pressure field, with the spatial dependence on the vector $r=(x,y,z)$ shown explicitly, and $\gamma(r)$ is the object function, which will be reconstructed from the scattered data, and we have used the subscripts $\omega\theta$ to indicate the solution dependence upon frequency (ω) and direction of incident field (θ). We also derived the Green's function which incorporates the special structure of the ambient media, i.e. $G_\omega(|r - r'|)$. The discretized version of this Green's function is obtained by convolution with the "sinc" basis functions: $\operatorname{sinc}(x) \equiv \sin(\pi x) / \pi x$ in the horizontal direction and "tent" functions in the vertical direction: $\Lambda(z) \equiv 1/\delta \begin{cases} \delta + z & -\delta \le z \le 0 \\ \delta - z & 0 \le z \le \delta \end{cases}$. denoted $G_\omega(|n|\delta)$, for $n=-nx,..0, .,nx-1$. The Green's function for the layered environment is broken up into a correlational and a convolutional part: $G_\omega(|n|\delta) \equiv G_V(|n|\delta) + G_R(|n|\delta)$ For n=0 $G_V(0) = \int_{u=0}^{u_\delta} J_o(0)\frac{2C(u)f(u)}{i\omega b_{sc}(u)}du$, where u_δ is an upper bound introduced by the band-limiting effect of convolution with the sinc functions. Furthermore $f(u) = \mathbf{R}^-\mathbf{R}^+\{z_1 - 1\} + \left\{1 - e^{-i\omega b_{sc}\delta} z_1\right\}$, $\mathbf{R}^+$, and $\mathbf{R}^-$ are generalized reflection coefficients from layers above and below the scattering object, $z_1 \equiv \left(e^{i\omega b_{sc}\delta} - 1\right) / i\omega b_{sc}\delta$, and $C(u) \equiv \left(1 - \mathbf{R}^-\mathbf{R}^+\right)^{-1}$. For n≠0, $G_V(|n|\delta) = \int_{u=0}^{u_\delta} J_o(u\omega|n|\delta) 2D(u)\frac{g(|n|\delta)}{\omega^2 b_{sc}^2 \delta}du$ where $D(u) \equiv \left(1 - \mathbf{R}^-\mathbf{R}^+\right)^{-1}\{1 - \cos(\omega b_{sc}\delta)\}$, and $g(|n|\delta) \equiv \left\{e^{-i\omega b_{sc}|n\delta|} + \mathbf{R}^-\mathbf{R}^+ e^{i\omega b_{sc}|n\delta|}\right\}$.

The correlational part is very similar, however, it is stored in a different order so that the correlation can be applied via a FFT. The discretized integral equations are used in an inversion procedure as follows. It may be the case that we have limited views available to us due to experimental limitations. If so we must rely upon multiple frequencies to increase the well posedness of the problem. To solve the multiple view and multiple frequency problem we minimize the residual between the predicted and observed fields at the receiver positions. This minimization can be carried out with either a nonlinear conujugate gradient procedure, or by an incomplete Newton method. The forward problem for an incident field in the direction θ and frequency ω is denoted by $\phi_{\theta\omega}$. This operator yields the scattered field at all receiver positions, given the incident field at angle θ, frequency ω, and the present estimate for the object function γ. $\phi_{\theta\omega} : \gamma \to \mathbf{f}_{\theta\omega}$. It turns out to be advantageous to attempt to solve the holomorphic (in γ) system

$$\mathbf{r}_{\theta\omega}^{(n)} \equiv \phi_{\theta\omega}\left(\gamma^{(n)}\right) - \mathbf{d}_{\theta\omega} = \mathbf{0} \quad \forall \begin{array}{l} \theta = 1, \ldots, \Theta \\ \omega = 1, \ldots, \Omega \end{array}$$

via a Newton Raphson procedure, where, as usual, the θ refers to the multiple views and the ω to the multiple frequencies available to us. That is, we assume that the noise level in the system is zero. We apply the Newton-Raphson procedure to this problem with the knowledge that applying a Gauss Newton algorithm to the associated least squares problem would give the same result. This leads to the overdetermined system

$$\left(\frac{\partial \phi_{\theta\omega}}{\partial \gamma}\right) \cdot \delta\gamma^{(n)} \approx -\mathbf{r}_{\theta\omega}^{(n)} \quad \forall \begin{array}{l} \theta = 1, \ldots, \Theta \\ \omega = 1, \ldots, \Omega \end{array}.$$

which must be solved for the γ-update $\delta\gamma$. Again, we can use the *complex analytic* version of the Hestenes overdetermined conjugate gradient algorithm, adapted for least squares problems to iteratively solve this system. This is equivalent to finding the minimum norm solution of the equations:

$$\left(\frac{\partial \phi_{\theta\omega}}{\partial \gamma}\right) \cdot \delta\gamma^{(n)} \approx -\mathbf{r}_{\theta\omega}^{(n)} \quad \forall \begin{array}{l} \theta = 1, \ldots, \Theta \\ \omega = 1, \ldots, \Omega \end{array}$$

The formula for the Jacobian in the layered medium situation in the presence of multiple frequencies, is,

$$\left(\frac{\partial \phi_{\theta\omega}}{\partial \gamma}\right) = \left(\left(\mathbf{I} - \mathbf{G}_\omega \cdot [\gamma]\right)^{-1}\left(\mathbf{G}_\omega \cdot [\mathbf{f}_{\theta\omega}]\right)\right)$$

where G_ω is now the *Layered* Green's function for the frequency ω (for simplicity of notation – which would otherwise get out of hand) I will suppress the L superscript on the Layered Green's function in the above and that which follows. Therefore, in effect, to determine the γ-update, $\delta\gamma$, we merely solve the multiple view problem for each particular frequency, that is, we solve the overdetermined system:

$$\left(\left[\mathbf{I} - \mathbf{G}_k \cdot [\gamma]\right]^{-1} \mathbf{G}_k\right) \otimes \mathbf{I}_{\Theta\times\Theta} \begin{bmatrix} [\mathbf{f}_{1k}] \\ \vdots \\ [\mathbf{f}_{\Theta k}] \end{bmatrix} \delta\gamma^{(n)} = -\begin{bmatrix} \left[\mathbf{r}_{1k}^{(n)}\right] \\ \vdots \\ \left[\mathbf{r}_{\Theta k}^{(n)}\right] \end{bmatrix},$$

which in component form is:

$$\left[\mathbf{I} - \mathbf{G}_\omega \cdot [\gamma]\right]^{-1} \mathbf{G}_\omega [\mathbf{f}_{\theta\omega}] \delta\gamma^{(n)} = -\left[\mathbf{r}_{\theta\omega}^{(n)}\right]$$

We end the presentation with several reconstructions based on computer generated data. The accuracy of the forward problem is guaranteed by comparison with several Bessel function expansion solutions derived analytically. We show that this method is very effective for propagating fields over large distances in layered environments to a computational grid that contains a complicated scattering object. The reconstructions are carried out using an efficient conjugate gradient and Newton based optimization method. The numerical efficiency of the method is preserved despite the presence of arbitrary layers both above and below the scattering grid by the break-up of the Green's function into correlational and convolutional parts, and the careful use of the Fast Fourier Transform. Also we discuss the various advantages and disadvantes of the method in various geometries. It is compared with the Split step Fourier method, and Finite Difference Time domain methods.

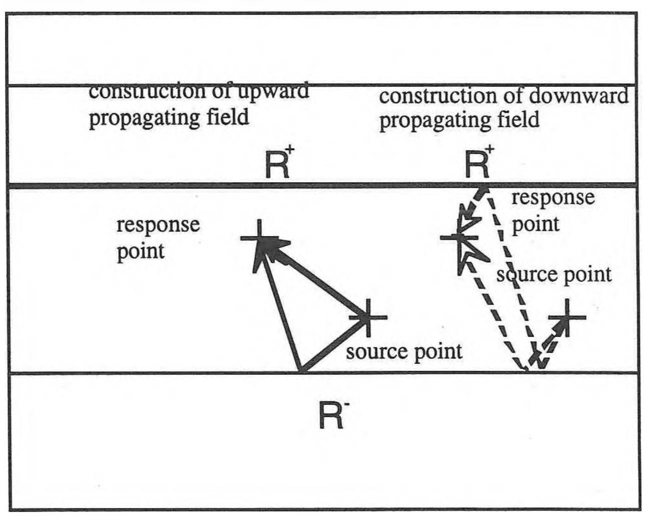

Figure 1

[1] Wiskin, J. W., Borup, D.T., Johnson, S. A., "Inverse scattering from arbitrary two-dimensional abjects in stratified environments via a Green's operator", *JASA 102, No. 2, Pt.1, pp. 853-864*, Aug. 1997