A 2-D Description of 3-D Effects in Sound Propagation

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Introduction

Mathematical modeling of low-frequency acoustic fields in the three-dimensionally inhomogeneous ocean is a computationally-intensive problem that remains intractable unless certain approximations are made to substitute the wave equation with a parabolic equation or to justify a reduction of the original problem to a sequence of 1-D and/or 2-D problems.

The adiabatic approximation [1, Sect. 7.1.3] is frequently used when modal concepts are applied to modeling and interpretation of the underwater acoustic fields in a range-dependent waveguide. For the 3-D problem of sound propagation in horizontally-inhomogeneous waveguides with gradual variation of the environmental parameters in the horizontal plane, the adiabatic mode method transforms into so-called "vertical modes - horizontal rays" approximation. Within this approximation, individual modes propagate from the CW source, without coupling, along certain curves in the horizontal plane. The curves are known as horizontal (or modal) rays, and their shape depends on sound frequency, mode order, and horizontal gradients of environmental parameters. Modal phase is given by an integral of the mode wavenumber along the horizontal ray, while modal amplitude depends on the variation of the cross section of a ray tube of the horizontal rays [1, Sect. 7.2].

It is typical to make a further approximation and substitute modal rays by radials from the source to receivers. Such an approximation is motivated by two facts. First, environmental parameters are often measured only along a radial. Second, modal ray curvature is normally small compared to the reciprocal of propagation range. The straight-ray approximation makes numerical tracing of modal rays unnecessary. It greatly reduces computational load and is especially important, if not imperative, for modeling broad-band fields and/or solving inverse problems in 3-D.

Although the straight modal ray approximation is implicit in the bulk of applications, its domain of validity has not been established. Moreover, it remains an open question what is the right way to calculate the modal amplitude along the radial. An expression originally proposed by Pierce [2] has been criticized as inconsistent with the reciprocity principle [3, 4]. Other expressions for the amplitude were put forward in [1, Sect. 7.2.2; 3; 4]. In this paper, we apply a perturbational analysis of horizontal ray equations to study accuracy of the straight modal ray approximation and to systematically derive an expression for the adiabatic mode amplitude that is more accurate than the *ad hoc* expressions proposed earlier in the literature.

Acoustic field in a horizontally-inhomogeneous waveguide

Consider acoustic field at a point with the horizontal coordinates $(x_2, y_2) = \boldsymbol{r}_2$ and the vertical coordinate z_2 due to a point source at $(\boldsymbol{r}_1, z_1)$. Assuming the unit strength of the source and slow, gradual dependence of environmental parameters on horizontal coordinates, one has [1, Sect. 7.2]

$$p(\boldsymbol{r}_2, z_2; \boldsymbol{r}_1, z_1) = \sum_n \left(\frac{1}{8\pi D_n} \right)^{1/2} f_n(z_1; \boldsymbol{r}_1) \times f_n(z_2; \boldsymbol{r}_2) \exp\left(i \int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} q_n ds - \frac{3\pi i}{4} \right). \quad (1)$$

Here p stands for acoustic pressure, f_n is a normalized shape function of a local mode of the order n, $q_n = q_n(\boldsymbol{r})$ is wavenumber of the mode. Integration in the exponent is along a horizontal ray $\boldsymbol{r}=\boldsymbol{r}(\tau, \psi_1)$ that connects the source and receiver, ψ_1 is the launch angle of the ray, i.e. the angle the ray makes with Ox coordinate axis at the source, τ is a parameter that defines a point along given ray. The quantity D_n in (1) is related to cross section area of a modal ray tube and can be calculated as follows:

$$D_n \equiv \frac{\partial(x, y)}{\partial(\tau, \psi_1)} = q_n(\boldsymbol{r}_2) \left[\left(\frac{\partial y}{\partial \psi_1} \right)_\tau \cos\psi_2 - \left(\frac{\partial x}{\partial \psi_1} \right)_\tau \sin\psi_2 \right]. \quad (2)$$

where ψ_2 stands for the ψ value at the receiver. Equations (1), (2) are exact within the adiabatic approximation but their application requires knowledge of q_n as a function of the 2-D vector $\boldsymbol{r}$ and an extensive ray tracing in the horizontal plane.

A simpler expression for the field was first proposed in the pioneering paper [2] by *Pierce*. It differs from (1) in that q_n is integrated along a radial from the source to receiver and D_n is calculated as

$$D_n = q_n(\boldsymbol{r}_2)|\boldsymbol{r}_2 - \boldsymbol{r}_1|. \quad (3)$$

It can be easily verified that (3) and the exact equation (2) are equivalent in the special case of cylindrically-symmetric medium with acoustic source located on the vertical axis of symmetry. Generally, however, D_n (3) is not invariant with respect to interchange of source and receiver positions, and therefore the resulting expression for acoustic pressure violates the reciprocity principle [1, Sect. 4.2]. To correct this shortcoming, some authors suggest using the expression [3]

$$D_n = \int_{x_<}^{x_>} q_n dx \quad (4)$$

instead of (3). To simplify notation, we chose here the coordinate system in such a way that $y_1=y_2=0$. Then integral in (4) is along an interval on the Ox axis; $x_< = \min(x_1, x_2)$, $x_> = \max(x_1, x_2)$. *Brekhovskikh* and *Godin* [1, Sect. 7.2.2] and independently *Porter* [4] proposed another expression:

$$D_n = q_n(x_1) q_n(x_2) \int_{x_<}^{x_>} \frac{dx}{q_n} \quad (5)$$

as an alternative to (3). This expression for D_n is manifestly reciprocal and reduces to the exact result (2) in the special case when environmental parameters are independent of the cross-range Cartesian coordinate y.

Qualitatively, for the expressions (3) - (5) to be useful when applied to a horizontally-inhomogeneous waveguide, true horizontal rays should be close to straight lines. However, the error introduced by using either (3) - (5) instead of (2) in the general 3-D environment has not been quantified in the literature.

Results of a perturbational analysis

To formalize the notion of "almost straight" modal rays, we represent the modal wavenumber squared as

$$q_n^2(\mathbf{r}) = k_0^2 + \epsilon g(\mathbf{r}), \quad |g|/k_0^2 \le 1, \quad 0 \le \epsilon \ll 1,$$
$$L_x = k_0^2/\|\partial g/\partial x\|, \quad L_y = k_0^2/\|\partial g/\partial y\| . \qquad \textbf{(6)}$$

Here L_x and L_y are representative spatial scales of the wavenumber variation in the range and cross-range directions. The parameter ϵ describes deviation of the horizontally-inhomogeneous ocean considered from a layered medium. The deviation is assumed small. When $\epsilon=0$, the environmental parameters depend on depth z only, and all modal rays are straight lines. When $\epsilon>0$, each modal ray has a nonzero curvature unless ∇g is tangent to the ray. The curvature is small as long as $\epsilon \ll 1$.

For the media described by (6), solutions to the differential equations governing modal rays [1, Sect. 7.2.1] can be found in terms of series in powers of ϵ. At the next step, eigenrays are found analytically for given source and receiver locations. Then, mode phase and amplitude are calculated. It turns out that all the three *ad hoc* expressions (3) - (5) lead generally to $O(\epsilon)$ errors in mode amplitude. That is, (3) - (5) are not generally any more accurate than straightforward approximations $D_n = k_0 |x_2 - x_1|$ or $D_n = q_n(\mathbf{r}_3)|x_2 - x_1|$, where $\mathbf{r}_3$ is an arbitrary point between the source and the receiver.

A more accurate result is given by

$$D_n = q_n(x_1,0)\, q_n(x_2,0) \left[\int_{x_<}^{x_>} \frac{dx}{q_n(x,0)} - \int_{x_<}^{x_>} dx\,(x - x_<)(x_> - x) \frac{\partial^2}{\partial y^2} \frac{1}{q_n(x,0)} \right] + O(\epsilon^2) . \qquad \textbf{(7)}$$

Among the infinite number of approximations to D_n that have accuracy $O(\epsilon^2)$, we have chosen the one that becomes *exact* in two important special cases of translational and rotational symmetry. Equation (7) reduces to (5) and is equivalent to (2) when the environmental parameters do not depend on the cross-range y. The error term in (7) is also zero when the environment is cylindrically-symmetric and the Ox axis (i.e., the horizontal line connecting source and receiver locations) intersects the vertical axis of symmetry. In the particular case of the source lying on the axis of symmetry, (7) reduces to (3).

The Fermat principle guarantees that the integral of q_n along a straight line between the source and receiver gives mode phase $\theta(\mathbf{r}_2, \mathbf{r}_1)$ to within $O(\epsilon^2)$. A second-order phase correction arises due to the horizontal ray curvature. The perturbation theory enables one to represent the correction explicitly:

$$\theta(\mathbf{r}_2, \mathbf{r}_1) = \int_{x_<}^{x_>} q_n(x,0)\,dx - \frac{1}{2\sqrt{q_n(x_1,0)\, q_n(x_2,0)}} \times \int_0^{|x_2 - x_1|} \frac{ds}{s^2} \left(\int_0^s \frac{\partial q_n}{\partial y}(x_< + a, 0)\, a\, da \right)^2 + O(\epsilon^3) . \qquad \textbf{(8)}$$

The result is invariant with respect to interchange of the source and receiver positions. To determine the mode phase with (8), one needs to know only mode wavenumber and its cross-range derivative along a radial from the source to receiver. The derivative can be easily calculated provided local mode shape functions as well as horizontal environmental gradients are known. An explicit expression for ∇q_n in rather general fluid and fluid/anisotropic solid waveguides can be found in [5]. It is essentially a weighted sum of (i) slopes of ocean floor and internal interfaces within ocean bottom, (ii) horizontal gradients of sound speed and density in water, and (iii) horizontal gradients of elastic parameters of the bottom.

For the perturbation theory to be valid, excursion of the horizontal eigenray from Ox axis should be small compared to L_y. In terms of the propagation range $R = |x_2 - x_1|$, this condition can be written as

$$\epsilon R^2 L_y^{-2} \ll 1 . \qquad \textbf{(9)}$$

Effects of cross-range variation of environmental parameters on modal amplitude and phase enter (7) and (8) through terms with derivatives of q_n with respect to y. The effects may be very significant under the condition (9). In particular, deviation of the mode phase from the first term on the right side of (8) can be large compared to unity. When the effects of horizontal ray curvature are negligible, one can disregard azimuthal coupling and substitute the 3-D problem by the 2-D problem of sound propagation in the vertical xz plane. From (7) and (8) it follows that the uncoupled azimuth approximation (sometimes also referred to as *N×2-D* approximation) is applicable as long as the inequality

$$10^{-2} k_0 \epsilon^2 L_y^{-2} R^3 \ll 1 , \qquad \textbf{(10)}$$

holds in addition to (9).

Implications the conditions (9) and (10) have on mathematical modeling of 3-D acoustic fields in horizontally-inhomogeneous deep- and shallow-water environments, will be discussed in the oral presentation.

Conclusion

The results obtained can be summarized as follows:

1. A new expression for the adiabatic mode amplitude in horizontally-inhomogeneous ocean is systematically derived that is manifestly reciprocal, requires knowledge of the local mode parameters only in the source-receiver vertical plane, and possess higher accuracy than previous formulations.
2. Adiabatic mode phase correction due to horizontal ray curvature is calculated analytically.
3. An applicability condition is obtained for the uncoupled azimuth approximation.

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