FORMULATION OF AN EQUATION OF DIFFUSION FOR HETEROGENEOUS RODS

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1. Introduction

The present article is devoted to the characterization of the vibratory behavior of heterogeneous structures, i.e when material or geometrical discontinuities are considered. Usual displacement formulations are rapidly limited when heterogeneity density and high frequency are considered. Energetic methods seem to be well adapted in this case, because an energetic response of the system is spatially smoother and more frequency robust than a classical displacement response.

Statistical energy Analysis (SEA) [1] is successfully used to characterize mean energies of lightly coupled subsystems; however the method is valid at high frequency when each mode of a subsystem is assumed to be uniformly probable over a frequency band. The energy flow method is based on the derivation of an equation of diffusion for homogeneous structures, in analogy with heat transfer [2]. An approximate solution is derived when quantities are spatially averaged over half a structural wavelength [3-4];. Difficulties however occur when spatially averaged energetic boundary conditions have to be specified, meaning that the method is not well adapted when heterogeneities occur.

In this paper, a new formulation characterizing the energetic diffusion in a rod with cross-sectional area discontinuities is presented. At low frequency, it appears that the energy flow is governed by heterogeneities since an heterogeneous gradient exists. The theoretical result is validated by a numerical simulation for a clamped-free heterogeneous rod which is excited at the free end by a power source.

2. Theoretical analysis

The structure under study is shown on Figure 1: It is composed of N elements, element p has a length L_p and contains n_p homogeneous rod of random lengths $l_{p,i}$ and cross-sectional areas $S_{p,i}$, the index p,i meaning homogeneous rod i from element p. The mean potential energy density for element p is $<\overline{U}>_p$.

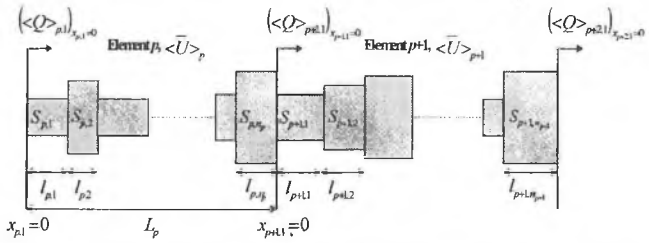

Figure 1. Description of two consecutive rod elements for random scheme.

The active power which is transmitted through an homogeneous rod p,i is expressed by

$$<Q>_{p,i}=\frac{1}{2}\mathrm{Re}\left\{\mathrm{j}\omega E S_{p,i}\frac{\partial u_{p,i}}{\partial x}u^*_{p,i}\right\}, \qquad (1)$$

where E is the complex Young's modulus and $u_{p,i}$ is the longitudinal displacement which is solution of the equation of motion for homogeneous rod outside of the external force points [5].

Displacement and axial force continuity between homogeneous rods p,i and p,i-1 [5] enforces power continuity

$$\left(<Q>_{p,i}\right)_{x_{p,i}=0}=\left(<Q>_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}, \qquad (2)$$

as well as the continuity of the second derivative if one derives equation (1) twice.

$$\left(\frac{\partial^2}{\partial x^2}<Q>_{p,i}\right)_{x_{p,i}=0}=\left(\frac{\partial^2}{\partial x^2}<Q>_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}. \qquad (3)$$

Similarly, equation (1) yields the following discontinuity condition for the first derivative of the power

$$\left(\frac{\partial}{\partial x}<Q>_{p,i}\right)_{x_{p,i}=0}=\frac{S_{p,i-1}}{S_{p,i}}\left(\frac{\partial}{\partial x}<Q>_{p,i-1}\right)_{x_{p,i-1}=l_{p,i-1}}. \qquad (4)$$

Let us define the first semi-local derivative of the power evaluated at the junction between elements p-1 and p

$$\left(\overline{\frac{\partial}{\partial x}}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left[\left(<Q>_{p+1,1}\right)_{x_{p+1,1}=0}-\left(<Q>_{p,1}\right)_{x_{p,1}=0}\right], \qquad (5)$$

and the second semi-local derivative

$$\left(\overline{\frac{\partial^2}{\partial x^2}}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left[\left(\overline{\frac{\partial}{\partial x}}<Q>_{p+1,1}\right)_{x_{p+1,1}=0}-\left(\overline{\frac{\partial}{\partial x}}<Q>_{p,1}\right)_{x_{p,1}=0}\right]. \qquad (6)$$

Assuming that dimensions of elements are small, and that the power slowly varies over each element (i.e. the structural wavelength is large compared to element lengths), then Taylor series expansions result in

$$\left(\overline{\frac{\partial}{\partial x}}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left(\sum_{r=1}^{n_p}l_{p,r}\frac{S_{p,1}}{S_{p,r}}\right)\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}. \qquad (7)$$

and

$$\left(\overline{\frac{\partial^2}{\partial x^2}}<Q>_{p,1}\right)_{x_{p,1}=0}=\frac{1}{L_p}\left[\frac{1}{L_{p+1}}\left(\sum_{r=1}^{n_{p+1}}\frac{l_{p+1,r}}{S_{p+1,r}}\right)\left(\sum_{r=1}^{n_p}l_{p,r}S_{p,r}\right)\left(\frac{\partial^2}{\partial x^2}<Q>_{p,1}\right)_{x_{p,1}=0}-S_{p,1}G_p\left(\frac{\partial}{\partial x}<Q>_{p,1}\right)_{x_{p,1}=0}\right]. \qquad (8)$$

where the heterogeneous gradient is defined by

$$G_p = \gamma_p - \gamma_{p+1} \tag{9}$$

from the heterogeneous density of element p

$$\gamma_p = \frac{1}{L_p} \sum_{r=1}^{n_p} \frac{l_{p,r}}{S_{p,r}} . \tag{10}$$

If one supposes that the heterogeneous gradient is different from zero, it is possible to neglect the first term in the right-hand side of equation (8) compared to the second term. This means that the second derivative of the power does no diverge in space. This argument is detailed below.

According to the local energy balance for steady state and outside of the input power points [4], the mean potential energy density of element p is defined by

$$< \overline{U} >_p = -\frac{1}{2\omega\eta} \left(\overline{\frac{\partial}{\partial x}} < Q >_{p,1} \right)_{x_{p,1}=0} \tag{11}$$

and the semi-local energetic gradient by

$$\overline{\nabla}_x < \overline{U} >_p = \frac{1}{L_p} [< \overline{U} >_{p+1} - < \overline{U} >_p] . \tag{12}$$

Using equation (7), it is then possible to formulate an equation of diffusion, expressing the energy flow from mean potential energy density

$$\overline{\nabla}_x < \overline{U} > + \alpha_r^{het} < \overline{U} >= 0 , \tag{13}$$

where

$$\alpha_r^{het} = \frac{G_p}{L_p \gamma_p} \tag{14}$$

is the heterogeneous coefficient of diffusion. Assuming a spatially-uniform coefficient, the solution of equation (13) is

$$< \overline{U} >= A \exp(-\alpha_r^{het} x) \tag{15}$$

from constant A, identified from boundary conditions. Potential energy density increases as the heterogeneous density does. The solution given by equation (15) stays admissible, as we saw, if the second derivative of the power is not great compared to first derivative. That means that the energy flow provided by loss factor, that is the first term of equation (8), goes in the same direction than the energy flow expressed by equation (13).

3. Numerical results

Let us consider the clamped-free heterogeneous rod illustrated on Figure 2, excited by a known input power $\left(< Q >_{1,1}\right)_{x_{1,1}=0} =< Q >_{in}$.

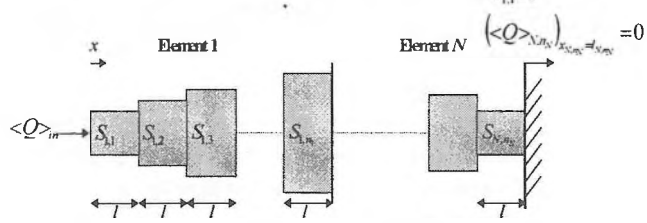

Figure 2. Free-clamped excited heterogeneous rod.

It is composed of N=400 elements, each of them containing n=4 homogeneous rods of random cross-sectional areas, and is excited at a frequency f=1000Hz. The Young's modulus is E_R=2.1×10^{11}Pa, the loss factor η=0.01 and the density ρ=7800kg/m^3, the lengths of homogeneous rod are similar, $l = 10^{-2}$ m. In order to validate the equation (13), the energy flow due to heterogeneities must decay in the same way as the energy flow due to loss factor, therefore one fixes constant decrease of the heterogeneous density along the length of the rod. The value of the energetic gradient theoretically predicted by equation (13) is compared to the numerical value derived from the exact equations on Figure 3.

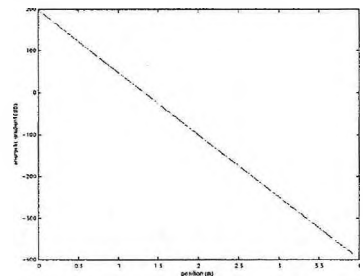

Figure 3. Semi-local energetic gradient :low frequency results, numerical value (—), theoretical solution (···).

The theoretical formulation is very well validated as the two curves fit perfectly.

4. Conclusion

In this work, a new formulation characterizing the energy flow in heterogeneous rod was derived when heterogeneities are modeled by cross-sectional area discontinuities. This method is semi-local when energetic diffusion is identified between elements containing a random scheme of heterogeneities. The energetic diffusion is governed at low frequency by cross-sectional area discontinuities as an heterogeneous gradient occurs. The theoretical results succeed in estimating the exact value provided by the numerical simulation when the energetic gradient is spatially plotted along the length of the rod.

A next important development in analyzing the energetic diffusion in heterogeneous structures is to study more complicated structures, such as beams with cross-sectional area discontinuities and plates with local masses. A long-term objective is to predict vibratory behavior of industrial structures such as car frames. It involves another important study when identifying energetic boundary conditions for heterogeneous coupled subsystems, that is local transmitted powers and energies.

5. References

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