DUCT MODES CONVERSION DUE TO THE PRESENCE OF LINERS

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ABSTRACT

This paper deals with the classical problem of transmission-reflection and modes conversion due to the presence of acoustic liners in duct. The model consists of an infinite rectangular duct with an unlined and lined section. The approach is based on the modal calculation of the total acoustic sound power in a duct by the determination of the modal coefficients of the transmitted waves at the impedance discontinuity junction. For a given propagating mode (plane wave or higher order mode), incident from the rigid duct, the total sound power is calculated in the lined section by summing the acoustic power over all generated modes. The example studied here shows how the conversion affects the nature of the incident mode as a function of the admittance of the liner, frequency and mode number. It also allows for the quantification of the attenuation provided by an acoustic duct liner.

SOMMAIRE

Le recours a des revêtements de parois absorbants le long d'un conduit est un moyen naturel de réduire le bruit généré par des machines tournantes (turboréacteurs, circuits de ventilation, etc.). D'un point de vue scientifique, ce contexte pose plusieurs problèmes fondamentaux tant sur le plan de la compréhension des phénomènes que sur celui de leur modélisation. Il faut en effet pouvoir préciser les aspects liés à la propagation guidée en présence de parois absorbantes. Cet article s'inscrit dans ce contexte et discute les effets liés à la conversion des modes qui est causée par la présence d'une discontinuité d'impédance en conduit. Pour ce faire, un model de conduit infini tridimensionnel a été étudié ou une partie est traitée par un revêtement acoustique. L'approche est basée sur un calcul modal de la puissance acoustique en conduit après détermination des coefficients modaux des modes transmis. Pour un mode de propagation donné (ondes planes ou modes supérieures), dans la section rigide du conduit, la puissance acoustique totale est calculée dans la section traitée en sommant les puissances des modes générés par la discontinuité d'impédance. L'exemple étudié ici montre comment cette conversion affecte la nature du mode incident en fonction de l'impédance du revêtement acoustique, de la fréquence et du mode. Il permet aussi de calculer l'atténuation apportée par un revêtement acoustique en conduit.

1. INTRODUCTION

The presence of acoustic treatments inside a duct induces a discontinuity problem which has been subject for many studies and researches in the 70's. One of the first complete studies in the subject was undertaken by Zorumski [1] who solved the case of a lined duct with a known acoustic impedance. Later on, Lansing et al. [2] studied the effect of the impedance of the duct walls on the transmission-reflection coefficients and on the radiation from the end of a baffled duct. Koch [3] used the Wiener-Hopf technique to study the effect of a finite layer of an acoustic material in a two dimensional duct on the propagation of the acoustic modes. He found that the acoustic field had considerably changed because of the conversion of the modes due to the presence of the liner. The above studies mainly dealt with semi-infinite duct having a simple geometry. The difficulties arise when the geometry or the shape of the duct is no longer straight due to the analytical nature of these developments. The present study offers the advantage of considering both the plane waves and the higher modes that propagate in the duct. It also enables the calculation of the attenuation from the transmitted modal sound power of each generated mode at the impedance junction.

The model consists of an infinite rectangular duct with an unlined and lined section. This model was chosen, even though complicated due to the presence of the lined section, as it seems to provide more realistic results of the attenuation provided by an acoustic liner, compared to the case of fully lined duct. The presence of a lined section in duct induces a discontinuity problem that is solved here. When a propagating acoustic duct mode arrives at an impedance discontinuity, it is partly transmitted into the lined section as a series of modes and partly reflected back.

The modal coefficients of the transmitted modes are determined by solving a set of modal equations grouped in a matrix form. The total modal sound power is calculated after the determination of the modal coefficients of the transmitted modes at the impedance discontinuity junction. For a given propagating mode (plane wave or higher order mode), incident from the rigid duct, the total sound power is calculated in the lined section by summing the acoustic power over all generated modes. The example studied here shows how the conversion affects the nature of the incident mode in function of the admittance of the liner, frequency and the mode number. It also allows quantifying the attenuation provided by an acoustic duct liner.

2. THEORY

A modal theory of propagation and attenuation of sound in a three-dimensional rectangular duct has been fully described in ref. [4]. The two first sections of this paragraph give a summary of the sound propagation in unlined and lined duct with formulation to calculate the modal coefficients of the transmitted modes.

The duct system studied here is shown in Figure 1.

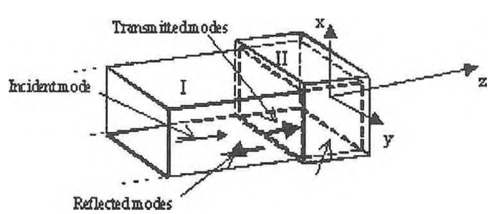

Figure 1: Three-dimensional infinite rectangular duct with an unlined and acoustically lined sections

The first section of the duct (section I) has rigid walls; the second section (section II) has all four walls treated with an acoustic liner with a known normalized acoustic admittance A. Let now consider an acoustic mode (m_0, n_0) incident from the rigid section I with a known amplitude $A_{m_0 n_0}$.

When the acoustic mode arrives at the admittance discontinuity junction, between the lined and unlined sections of the duct, it will be partly transmitted into the lined section as a series of modes with complex amplitudes $\hat{A}_{qp}$ and partly reflected back into the rigid section I with complex modal amplitudes $\hat{B}_{mn}$. The acoustic energy in the incident mode is thus partially transmitted into the lined section and partially reflected back. The modal coefficients of the transmitted waves are determined in section 2.2.

2.1 Propagation and attenuation in the duct

The acoustic field inside the duct is determined by solving the Helmholtz equation for the linear case and in the absence of sources and flow:

$$\Delta P + k^2 P = 0 \qquad (1)$$

where P is the acoustic pressure, $k = \omega / c_0$ is the wave number, ω the angular frequency and ρ, c_0 are the ambient density and speed of sound respectively. The sidewall boundary conditions are:

$$\frac{\partial P}{\partial \vec{n}} = \begin{cases} 0 & ; in\ section\ I \\ i k A P & ; in\ section\ II \end{cases} \qquad (2)$$

at $x = \pm l_x / 2$ and $y = \pm l_y / 2$

where A is the normalized wall admittance of a "locally reacting" boundary and $\vec{n}$ is the outward normal.

The general solution for the pressure field in sections I and II of the duct is given by,

$$P_I(x,y,z) = A_{m_0 n_0} \Psi(K_{m_0} x) \Psi(K_{n_0} y) e^{-i K_{m_0 n_0} z} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \hat{B}_{mn} \Psi(K_m x) \Psi(K_n y) e^{i K_{mn} z} \qquad (3)$$

where,

$$\begin{cases} \Psi(K_m x) = {}^{\cos}_{\sin}(K_m x) \\ \Psi(K_n y) = {}^{\cos}_{\sin}(K_n y) \end{cases} \qquad (4)$$

are the eigenfunctions. The term $e^{i\omega t}$ is implicit throughout the paper. A cosine function is used for the even modes and a sine function for the odd modes. The transverse wave numbers $K_m = (m-1)\pi / l_x$ and $K_n = (n-1)\pi / l_y$ are determined by the boundary conditions (2) and where l_x and l_y are the cross-sectional dimensions of the duct in the x and y direction respectively, and m, n are integers different from zero.

The axial wave number is given by the following dispersion equation $K_{mn}^2 = k^2 - (K_m^2 + K_n^2)$

The propagation of the waves in the axial direction is possible as long as the axial wave number $K_{mn}^2 > 0$. According to the dispersion equation, this is true for $\omega > c_0\sqrt{K_m^2 + K_n^2} = \omega_{mn}^c$. Below this "cut-off" frequency ω_{mn}^c, the axial wave number K_{mn} becomes a purely imaginary number, and the propagation factors in equation (4) turn into $e^{-|K_{mn}z|}$; which means the amplitudes of these modes decay exponentially with axial distance from the source: they are "cut-off". Note that the mode (m_0, n_0) is just one particular mode over all possible (m,n) modes.

The general solution, for the pressure field in Section II, of Helmohltz equation (1) with boundary conditions (2) is (assuming that reflections from the end of the duct are neglected)

$$P_{II}(x,y,z) = \sum_{q=1}^{\infty}\sum_{p=1}^{\infty} \hat{A}_{qp}\ \hat{\Psi}(\hat{K}_q\, x)\,\hat{\Psi}(\hat{K}_p\, y)\, e^{-i\hat{K}_{qp}z} \qquad (5)$$

where,

$$\begin{cases} \hat{\Psi}(\hat{K}_q\, x) = {}^{\cos}_{\sin}(\hat{K}_q\, x) \\ \hat{\Psi}(\hat{K}_p\, y) = {}^{\cos}_{\sin}(\hat{K}_p\, y) \end{cases} \qquad (6)$$

are the complexes eigenfunctions. The transverse wave numbers $\hat{K}_q = \hat{\mu}_q \pi / l_x$ and $\hat{K}_p = \hat{\mu}_p \pi / l_y$ are determined by the boundary condition (2). The axial wave number is given by the dispersion equation: $\hat{K}_{qp}^2 = k^2 - (\hat{K}_q^2 + \hat{K}_p^2)$, where $\hat{\mu}_q$ and $\hat{\mu}_p$ are complex numbers. Note, in this case, the "cut-off notion" has no physical meaning. Assuming $\hat{K}_{qp} = (\alpha \pm i\beta)k$, α is the non-dimensional axial wave number and β is the damping factor of the mode. β should be positive for a mode propagating in $z>0$ direction. This means that we should look for a solution of the dispersion equation that provides attenuation.

Solving equation (5) by the method of separation of variables and imposing the boundary conditions (2) leads to the following characteristic equations

$$(\hat{K}_e\, l_k/2)\ {}^{\tan}_{\cot}(\hat{K}_e\, l_k/2) = \pm i\, k\, A l_k/2 \qquad (7)$$

where the term in tangent is used for even modes and the one with cotangent for odd modes. The e and k indices represent q or p and x or y respectively depending on the propagation direction.

The axial and transverse wave numbers were computed using a numerical scheme developed in ref. [2, 3], where the characteristic equation is transformed into a first order non-linear differential equation. The differential equation is integrated using a Runge-Kutta algorithm with appropriate initial values. The transverse wave numbers then are used to compute the axial wave number using the dispersion equation.

2.2 Calculation of the Transmitted Modal Coefficients

The acoustic pressure P and the acoustic velocity V are related by the momentum equation

$$\nabla \overline{P} = -i\, k\, \rho\, c_0\, \overline{V} \qquad (8)$$

Using equations (3) and (5), the axial velocity in both sections (I and II) can be written as

$$\begin{cases} V_I(x,y,z) = (1/k\,\rho\, c_0)\left\{ A_{m_0n_0}\, K_{m_0n_0}\, e^{-iK_{m_0n_0}z}\,\Psi(K_{m_0}\, x)\,\Psi(K_{n_0}\, y) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\hat{B}_{mn}\, K_{mn}\, e^{iK_{mn}z}\,\Psi(K_m\, x)\Psi(K_n\, y)\right\} \\ V_{II}(x,y,z) = (1/k\,\rho\, c_0)\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\hat{A}_{qp}\,\hat{K}_{qp}\,\hat{\Psi}(\hat{K}_q\, x)\,\hat{\Psi}(\hat{K}_p\, y)\, e^{-i\hat{K}_{qp}z} \end{cases} \qquad (9)$$

The unknown amplitudes $\hat{A}_{qp}$ and $\hat{B}_{mn}$ in equations (3), (5) and (9) are determined from a system of linear equations obtained by applying the continuities conditions: the pressures and axial velocities in the two sections of the duct must be equal at the junction $z=0$ $P_I(x,y,0) = P_{II}(x,y,0)$ and $V_I(x,y,0) = V_{II}(x,y,0)$

Thus, by substituting equations (3), (5) and (9) into these two equations, and using the orthogonality properties of the eigenfunctions, the following system for the transmitted modal coefficients is obtained

$$\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\hat{A}_{qp}\,\hat{I}_{m'q}\,\hat{J}_{n'p}\,(\hat{K}_{qp} + K_{m'n'}) = A_{m_0n_0}\,\frac{(1+\delta_{m_01})}{2}\frac{(1+\delta_{n_01})}{2}\,\delta_{m_0m'}\,\delta_{n_0n'}\,(K_{m'n'} + K_{m_0n_0}) \qquad (10)$$

where δ is the Kronecker delta and

$$\hat{I}_{ij}=\frac{1}{l_x}\int_{-l_x/2}^{l_x/2}\Psi(K_i\,x)\,\hat{\Psi}(\hat{K}_j\,x)dx \quad (11)$$

$$\hat{J}_{ij}=\frac{1}{l_y}\int_{-l_y/2}^{l_y/2}\Psi(K_i\,y)\,\hat{\Psi}(\hat{K}_j\,y)dy \quad (12)$$

Equations (11) and (12) are solved analytically. The system indices, m and n, vary from 1 to **M** and 1 to **N** respectively, while q, p vary from 1 to **Q** and 1 to **P** respectively after truncation. Therefore, we have $[\mathbf{Q}.\mathbf{P}]$ complex equations and $[\mathbf{Q}.\mathbf{P}]$ complex unknown which are the transmitted modal coefficients. **M** and **N** are the total propagating modes in a duct following each transverse direction. The final linear system (10) takes the final matrix form $[\mathbf{a}].[\mathbf{X}]=[\mathbf{b}]$

where,

- $[\mathbf{a}]$ complex vector which contains the modal transmitted coefficients to be determined,
- $[\mathbf{X}]$ complex matrix which depends on the modes (m,n) and on the eigenvalues of the system,
- $[\mathbf{b}]$ known vector which depends on the incident mode (m_0,n_0) and its amplitude $A_{m_0n_0}$.

The final matrix $[\mathbf{X}]$ is square and the dimension of the system is multiplied by 2 to account for the complex numbers, therefore the final matrix dimensions are $[2.\mathbf{Q}.\mathbf{P},2.\mathbf{Q}.\mathbf{P}]$. Further, the truncation is performed at $\mathbf{Q}=\mathbf{M}+2$ and $\mathbf{P}=\mathbf{N}+2$. This truncation was checked when calculating all possible transmitted and reflected coefficients at the discontinuity junction for any incident mode (m_0,n_0). The determination of the modal coefficients and sound powers of the transmitted modes will show that it's worthless and time consuming to consider a number of modes (generated in section II) greater than the limit chosen above. An LU decomposition algorithm with matrix inversion was used to solve the matrix system.

2.3 Sound power calculation

The modal axial acoustic velocity of a propagating mode, following $z>0$ direction, is given by:

$$\mathrm{V}_{mn}=\frac{1}{\rho\,c_0}\left(\frac{K_{mn}}{k}\right)P_{mn} \quad (13)$$

and the modal acoustic intensity is given by,

$$I_{mn}=\overline{P_{mn}\,V_{mn}}=\frac{1}{2}\mathrm{Re}\{P_{mn}\,V_{mn}^*\} \quad (14)$$

where V_{mn}^* denotes the complex conjugate of V_{mn}.

The sound power is obtained by integration of the intensity over the duct section (ref. [5, 6]):

$$\Pi=\iint_S I\,dS \quad (15)$$

where S is the duct cross section area.

Rigid duct case

For an incident mode with a given index (m,n), P_{mn} is given by the solution of the wave equation. The modal acoustic sound power in a rigid duct is:

$$\Pi_{mn}=\frac{1}{2}\left(\frac{S}{\varepsilon_m\varepsilon_n}\right)\left(\frac{1}{\rho c_0}\right)|A_{mn}|^2\left(\frac{K_{mn}}{k}\right) \quad (16)$$

with K_{mn} real, S the cross section area of the duct, and $\varepsilon_i=\begin{cases}1 & ;if\ i=1\\ 2 & ;if\ i>1\end{cases}$

Lined duct case

The modal acoustic sound power is obtained in similar way as for the rigid duct case. The modal sound power, for a given generated mode in section II, is given by:

$$\Pi_{qp}^{l}=\frac{1}{2}\left(\frac{1}{\rho\,c_0}\right)|\hat{A}_{qp}|^2\,I_q\,I_p\,\mathrm{Re}\left\{\frac{\hat{K}_{qp}}{k}\right\}e^{-2\mathrm{Im}(\hat{K}_{qp})z} \quad (17)$$

with

$$I_q=\int_{-l_x/2}^{l_x/2}{}^{\cos}_{\sin}(\hat{K}_q\,x)\,{}^{\cos}_{\sin}(\hat{K}_q^*\,x)dx \quad (18)$$

and

$$I_p=\int_{-l_y/2}^{l_y/2}{}^{\cos}_{\sin}(\hat{K}_p\,y)\,{}^{\cos}_{\sin}(\hat{K}_p^*\,y)dy \quad (19)$$

The integrals are calculated analytically by using the trigonometry functions' properties as follows:

$$\int_{-l_x/2}^{l_x/2}{}^{\cos}_{\sin}(\hat{K}\,x)\,{}^{\cos}_{\sin}(\hat{K}^*x)dx=\frac{1}{2}\left\{\frac{sh[\mathrm{Im}(\hat{K}l)]}{\mathrm{Im}(\hat{K}l)}\pm\frac{\sin[\mathrm{Re}(\hat{K}l)]}{\mathrm{Re}(\hat{K}l)}\right\} \quad (20)$$

The total transmitted sound power in region II is the sum of the modal acoustic power of each generated mode, and is given by:

$$\Pi_t^{II}=\sum_{q=1}^{\infty}\sum_{p=1}^{\infty}\Pi_{qp}^{II} \quad (21)$$

Finally, the expression of the total attenuation provided by the acoustic treatment over a certain length L is given by:

$$\Delta\Pi_t^{II}(dB)=10\log_{10}\left(\sum_{q=1}^{\mathbf{Q}}\sum_{p=1}^{\mathbf{P}}\Pi_{qp}^{II}(L)\Big/\sum_{q=1}^{\mathbf{Q}}\sum_{p=1}^{\mathbf{P}}\Pi_{qp}^{II}(0)\right) \quad (22)$$

3. APPLICATION

An example of application, which deals with noise suppression from a turboshaft engine inlet, is discussed in this section. The aim is to attenuate the high frequency noise, generated by the compressor (fundamental mode of blade passing frequency), by lining the inlet with a composite material. The frequency of concerns is situated between 8 and 10 kHz. The liner consists of a solid backplate and a layer of feltmetal material separated by honeycomb structures. The normalized acoustic admittance of the used liner is referred to R1. It has been determined experimentally by using an impedance tube measurement.

Results from a two-dimensional case is shown in Figure 3 and Figure 4 where the total transmitted non-dimensional sound power $\left(\Pi_t^{II}/\Pi_1\right)$ for the least attenuated incident mode (fundamental mode) is plotted versus the frequency with a 500 Hz step for a weak and strong constant normalized admittance values $A=(.1,+.1)$ and $A=(1.,+.1)$ respectively, and at different location $z=0$ m (at the discontinuity junction), $z=0.1$ m and $z=0.2$ m. The duct width is 0.2 m, $\mathbf{M}=12$ at $f_{\max}=10\,kHz$, and the system truncated at $\mathbf{Q}=20$.

For a weak admittance value, the graph in Figure 3 at $z=0.1$ m, shows that the transmission is complete with almost no reflections. However for the same mode, Figure 4 shows a much stronger reflection at the discontinuity junction. The graphs of this figure exhibit a wavy behavior, especially at $z=0$ m, which due to the successive contribution of the reflected modes occurring at the discontinuity junction while the frequency increases.
The results from an incident mode 3 are shown in Figure 4 for a strong constant normalized admittance value.
It is clear that the sound power ratio increases with the frequency at a constant admittance value. This representation also allows to globally quantifying the attenuation provided by the liner.

A three-dimensional square duct of 0.1 m by 0.1 m cross section was considered in the calculation of the total transmitted power $\left(\Pi_t^{II}/\Pi_{m0n0}\right)$. Figure 5 and Figure 6 show the calculation results for modes (1,1) and (2,2) for different admittances (constant weak and strong admittance value and a variable admittance R1). The same behaviors as observed in the two-dimensional case described above occur here.

Finally, the graphs of Figure 7 and Figure 8 show the total modal attenuation in dB, provided by the liner, versus the lined length for different incident modes and at $K\,l=29.3$. The model duct had 0.1 m by 0.1 m cross section, and the matrix system has been truncated at $\mathbf{Q}=\mathbf{P}=20$ for the calculation of the transmitted coefficients, that means, almost a double number of modes in the lined section (section II) than the propagating modes in section I was considered at a maximum frequency of 10 kHz where $\mathbf{M}=\mathbf{N}=12$.
It can be seen that liner with normalized admittance R1 provides better attenuation in the frequency range of concerns.

4. CONCLUSION

The objective is to provide formulations that allow the quantification of the attenuation provided by a liner that has a known acoustic admittance. The modal analytical approach described here permits an understanding of the modes conversion phenomena. The total transmitted sound power has been calculated for different wall admittance values representing a weak, strong and optimal "R1" attenuation at different locations.

5. REFERENCES

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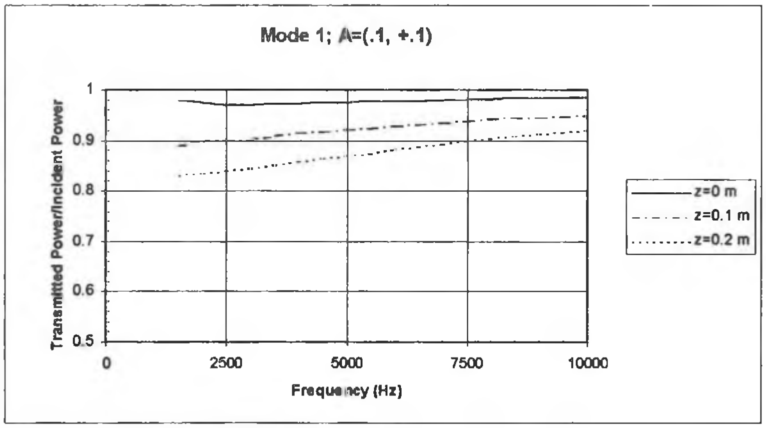

Figure 2: Total transmitted to incident power ratio $\left(\Pi_t^{II}/\Pi_{m0}\right)$ ***as function of frequency for incident mode 1 and a normalized acoustic admittance*** $A=(.1,+.1)$

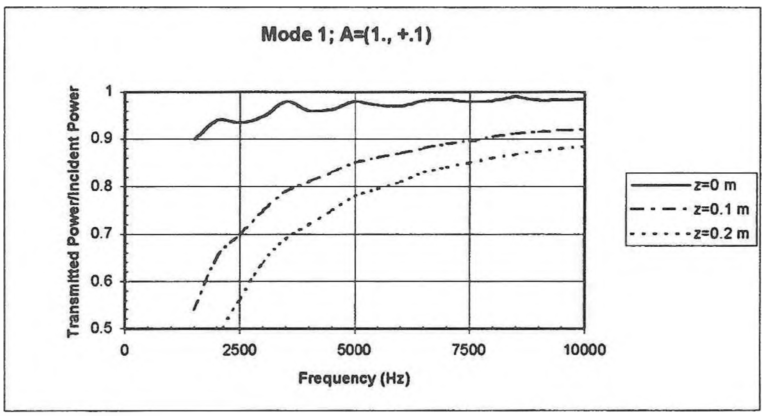

Figure 3: Total transmitted to incident power ratio $\left(\Pi_t^{II}/\Pi_{m0}\right)$ ***as function of frequency for incident mode 1 and a normalized acoustic admittance*** $A=(1.,+.1)$

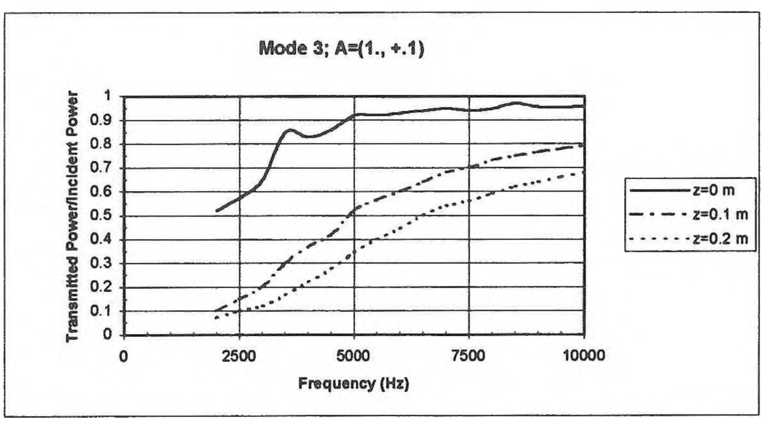

Figure 4: Total transmitted to incident power ratio $\left(\Pi_t^{II}/\Pi_{m0}\right)$ ***as function of frequency for incident mode 3 and a normalized acoustic admittance*** $A=(1.,+.1)$

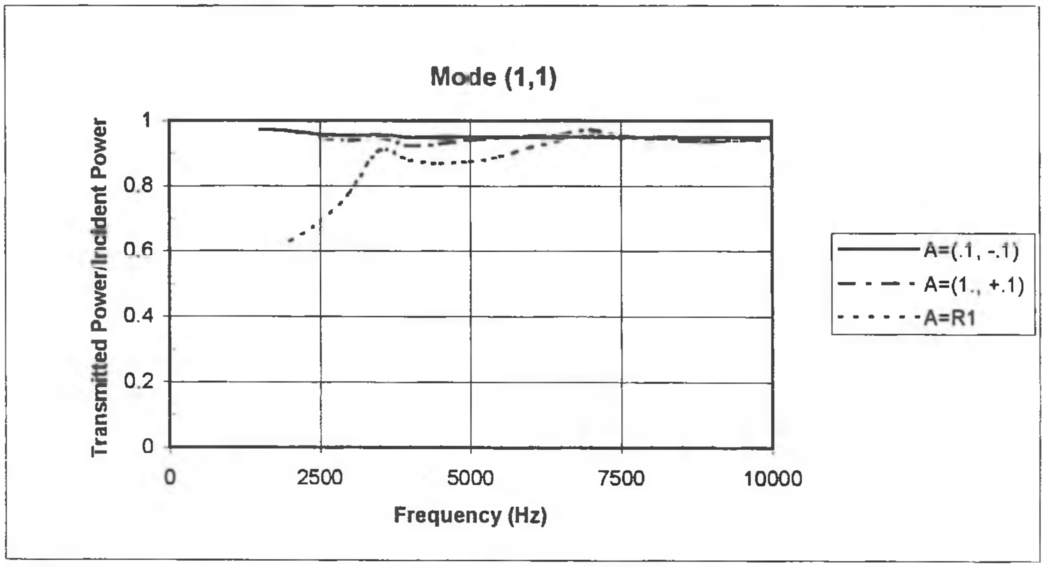

Figure 5: Total transmitted to incident power ratio $\left(\Pi_t^{II}/\Pi_{m0n0}\right)$ ***as function of frequency for incident mode (1, 1) and different normalized acoustic admittance values.***

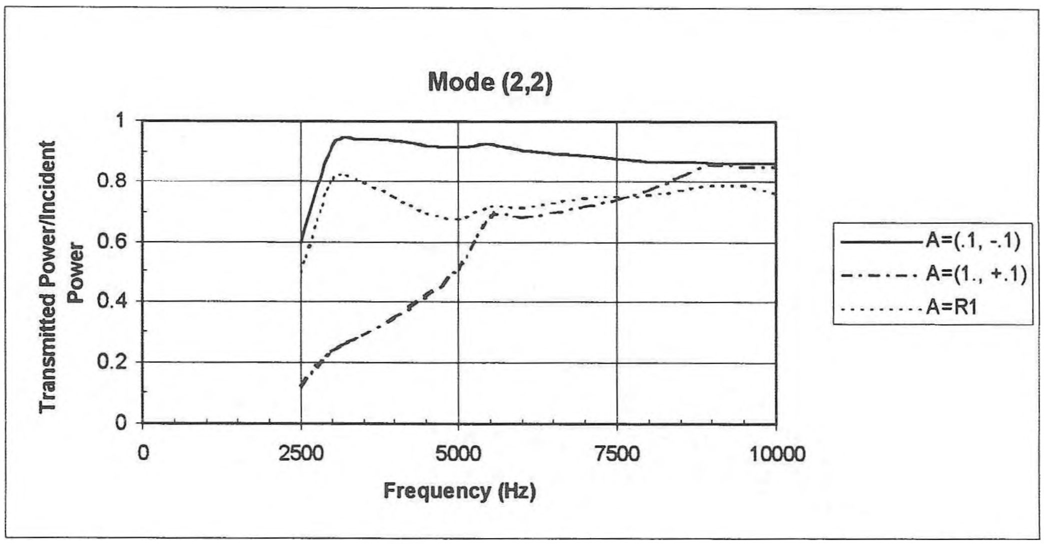

Figure 6: Total transmitted to incident power ratio $\left(\Pi_t^{II}/\Pi_{m0n0}\right)$ ***as function of frequency for incident mode (2, 2) and different normalized acoustic admittance values.***

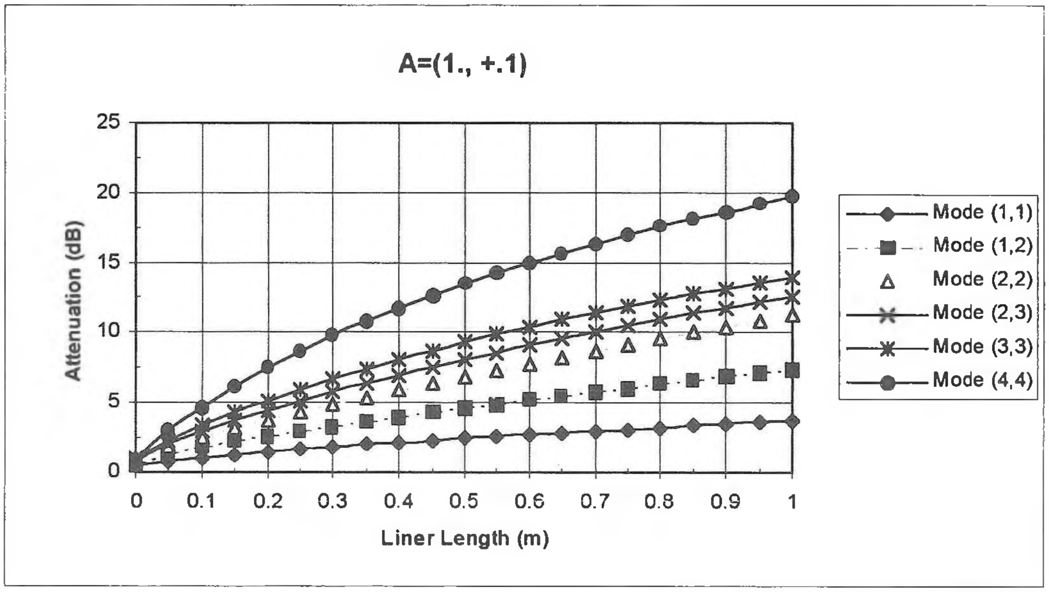

Figure 7: Modal attenuation $\left(\Pi_t^{II}/\Pi_i\right)$ *as function of liner length for different incidents modes and a normalized acoustic admittance* $A=(1.,+.1)$

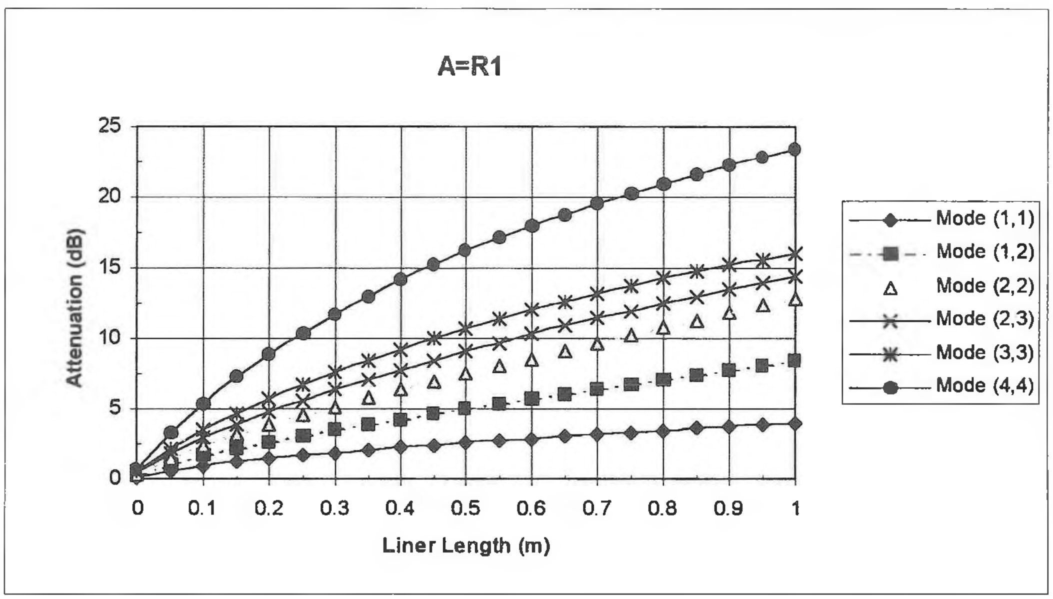

Figure 8: Modal attenuation $\left(\Pi_t^{II}/\Pi_i\right)$ *in function of the distance for different incidents modes and a normalized acoustic admittance* $A=R1$