A Method For Selecting Chief Points In Acoustic Scattering

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ABSTRACT

In this work, the nonuniqueness problem of solving surface integral equation of acoustic scattering is considered. The solution of the acoustic scattering integral equation is not unique at some frequencies. A unique solution can be obtained by adding some constraints to the problem at some interior points of the scatterer. The primary difficulty is the lack of formalized method for the selection on the interior points to guarantee uniqueness. A simplified method for selecting interior points for CHIEF method is proposed. The new augmented surface integral equation is successful in reducing the needed number of points to solve at the characteristic frequencies of the scattering problem where a unique solution does not exist. The implementation of the method exploits the earlier computations used in selecting the interior points. Numerical results are presented at some characteristic frequencies for an axisymmetric body. A comparitive analysis is also presented to evaluate the potential of the proposed method.

RÉSUMÉ

Ce travail, le problème de non unicité en résolvant l'équation intégrale extérieure de la dispersion acoustique est considérée. La solution de l'équation intégrale de dispersion acoustique n'est pas unique à quelques fréquences. Une solution unique peut être obtenue en ajoutant quelques contraintes au problème à quelques points intérieurs du diffuseur. La difficulté primaire est le manque de méthode formalisée pour le choix sur les points intérieurs pour garantir l'unicité. On propose une méthode simplifiée pour choisir les points intérieurs pour la méthode en CHIEF. La nouvelle équation intégrale extérieure augmentée, prouve le succès en réduisant le nombre nécessaire de points en cas de solution aux fréquences caractéristiques du problème de dispersion, auquel le problème n'a pas une solution unique. L'exécution de la méthode exploite les calculs précédents utilisés en choisissant les points intérieurs. Des résultats numériques sont présentés à quelques fréquences caractéristiques pour un corps axisymmetric. Une analyse de comparaison est également présentée pour évaluer le potentiel de la méthode proposée.

1. INTRODUCTION

Integral equation methods have been used to solve exterior acoustic radiation and scattering problems for many applications. In these problems, the external pressure is represented in terms of a distribution of an acoustic field on the surface of a scatterer or radiator. By forcing this representation to match a specified velocity distribution on the surface, an integral for the unknown source strengths was obtained. Once the source density is obtained, the pressure at any point in the exterior region can be computed. Integral equations of this type do not have a solution at the natural frequencies of an associated interior Dirichlet problem [1].

The surface Helmholtz integral equation is advantageous in that the problem's dimensionality is reduced by one and an infinite domain is transformed to finite boundaries in which the far-field radiation condition is satisfied. The solution of the acoustic (Helmholtz integral) boundary value problem is unique for all frequencies [1, 2]. However, the standard Helmholtz integral equation fails to yield unique solutions at the natural frequencies. For direct formulations, both the Dirichlet and Nuemann problems have the same characteristic frequencies as the eigen frequencies of the interior Dirichlet problem [5]. This problem is one of nonuniqueness rather than nonexistence [1]. Non-uniqueness is a purely mathematical problem arising from the breakdown of boundary integral representation rather than from the nature of the physical problem [5, 3]. For solving this problem, two main approaches have been followed; the first method is called CHIEF, the combined Helmholtz integral equation approach, which is perhaps the most widely used in engineering applications. The second method combines the Helmholtz integral equation with its normal derivative.

The CHIEF method uses the surface Helmholtz integral equation, combined with the corresponding interior Helmholtz integral equation, to form over-determined system of equations which can then be solved. This method may not function properly at the characteristic frequencies when some of the interior points coincide with a nodal surface of the related interior problems. More CHIEF points may be required to yield unique solutions at the characteristic frequencies. Therefore, in this method, the number and location of CHIEF points must be effectively selected, particularly in the high frequency range. The second method is severely limited in that the hyper-singular integral must be evaluated and so numerical difficulties arise. The tangential derivative formulation has been derived to regularize the highly singular kernels [4].

In general, a comparison of these two methods shows that they introduce their own particular complications [4]. Many enhancements have already been introduced to both these methods. Benthien and Schenck, in a recent survey, reviewed various methods for handling the nonexistence and non-uniqueness [1].

The CHIEF method is the most extensively used in engineering applications [5]. A potential problem with this approach is the choice of interior points for the supplemental equations. The selected interior points must not be a nodal point of the corresponding interior eigen mode. It has been demonstrated that it is only needed one non-nodal point (good) to establish a unique solution [2]. A survey on the location of "good" points is, also, found in Reference 2. The primary difficulty is the lack of formalized method for the selection on the interior points to guarantee uniqueness [6]. Despite that there is no systematic way to select the interior points in CHIEF method, the selection of effective interior points is not difficult. Several decades of practical experience has shown that effective interior points are not difficult to choose and that the CHIEF method is very robust [1].

In this work a monitoring method is proposed to test the effectiveness of interior points for CHIEF. The proposed method is simple, needs neither rigorous mathematical formulations nor significant computational burden. The proposed method exploits a non-unique solution at the required characteristic frequency on the scatterer surface. The non-unique solution, is, then used to compute the field at some interior points. The computed field strengths can monitor any nodal points using a simple criterion. The matrix used in the first run is, then, augmented with non-nodal interior points to give the over-determined system of equations which can be solved for the unique surface field.

2. INTEGRAL EQUATION DERIVATION

The governing equation for the propagation of acoustic waves through an unbounded homogenous medium is described by the wave equation

$$\nabla^2 \phi(r,t) = \frac{1}{c^2} \frac{\partial^2 \phi(r,t)}{\partial t^2} \quad (1)$$

where, ∇^2 denotes the Laplacian operator in three dimensions. ϕ the velocity potential at r and t. c is the speed of sound in the medium at the equilibrium state. The velocity potential ϕ is

$$\mathbf{u} = \nabla\phi \quad (2)$$

It is common practice to express the velocity potential as

$$\phi = \phi^i + \phi^s \quad (3)$$

where ϕ^i and ϕ^s are the incident and the scattered velocity potentials. The excess pressure can be written as

$$p = -\rho_o \frac{\partial\phi}{\partial t} \quad (4)$$

where ρ_o is the density of the fluid at the equilibrium state. It follows that,

$$p = p^i + p^s \quad (5)$$

where, p^i and p^s are the incident and the scattered pressures. The differential equation for time-harmonic waves with a time factor $e^{i\omega t}$ takes the form

$$(\nabla^2 + k^2)\phi = 0 \quad (6)$$

where, $k=\omega/c$ is the wave number, and ω the angular frequency. Accordingly, equation (4) becomes

$$p = -i\rho_o \omega\phi \quad (7)$$

At the surface of a hard scatterer, the normal component **u.n** of the fluid particle velocity **u** is zero; so

$$\frac{\partial\phi}{\partial n} = 0 \quad (8)$$

where **n** is the unit vector normal to the surface of the scatterer body and into the surrounding space, and n is the distance along the external normal vector **n.** At the surface of a soft scatterer, the excess pressure is zero, (i.e.),

$$\phi = 0 \quad (9)$$

Equations 8 and 9 represent the Neumann and Dirichlet boundary conditions of the differential equation, Equation 1, respectively.

The equivalent boundary integral formulation of Equation 6 is valid for an acoustic medium B' exterior to a finite body B with surface S and a unit normal **n,** pointing into B'. The body is submerged into an infinite linear acoustic medium. When a harmonic acoustic wave ϕ^i impinges upon the body *B,* the resulting integral equation for smooth boundaries has the following form.

$$C(P)\,\phi(P) = \int_S \left(\phi(Q)\,\frac{\partial \psi(P,Q)}{\partial n} - \psi(P,Q)\frac{\partial \phi(Q)}{\partial n} \right) dS_Q + 4\,\pi\,\phi^i(P) \tag{10}$$

Equation 10 is the surface Helmholtz integral equation, where, $\phi(P) = \phi(r_P)\, e^{i\omega t}$ at point *P* and *Q* is a point on the body surface.

The free-space Green's function ψ for the Helmholtz wave is given by

$$\psi(P,Q) = e^{-ikR} / R \tag{11}$$

where, *R* is the distance between the field point *P* and a source point *Q*, and ***n*** is the outward directed normal at *Q*. The distance is expressed, vectorially, as

$$R = \left| \boldsymbol{r_P} - \boldsymbol{r_Q} \right|, \tag{12}$$

where, $\mathbf{r_p}$ is the vector to point *P* from the origin and similarly for *Q*. The coefficient *C(P)* is defined at *P* on *S* provided that there is a unique tangent to *S* at such a *P*, as

$$C(P) = \begin{cases} 0 & for\ \ P \in B' \\ 4\pi & for\ \ P \in B \\ 2\pi & for\ \ P \in S \end{cases} \tag{13}$$

When *P* occupies a point on the surface *S*, there is no unique tangent plane (e.g., when *P* is on an edge of a sharp corner), *C(P)*, then relates to the solid angle α by [4].

$$\begin{aligned} C(P) &= 1 - \alpha / 4\pi \\ &= 4\pi + \int_S \frac{\partial}{\partial n}\left(\frac{1}{R(P,Q)} \right) dS_Q \end{aligned} \tag{14}$$

3. DESINGULARIZATION

We consider here only the fully axisymmetric scattering case, (i.e), both the body shape and the acoustic variables are independent of the angle of the revolution of the body. For scattering, this implies that the direction of the incident wave must coincide with the axis of revolution of the body. This simplification is used to test the proposed nonuniqueness solution. The singularity regularization is similar to that used in Seybert and Ranganathan [2]. This formulation is summarized in the next section.

For an axisymmetric body, the integrals in Equation 10 can be rewritten using a cylindrical coordinate system (ρ, θ, z) as

$$\int_L \phi(Q) \left[\int_0^{2\pi} \frac{\partial}{\partial n}\left(\frac{e^{-ikR(P,Q)}}{R(P,Q)} \right) d\theta(Q) \right] \rho(Q) dL(Q) \tag{15}$$

and

$$\int_L \frac{\partial \phi(Q)}{\partial n} \left[\int_0^{2\pi} \left(\frac{e^{-ikR(P,Q)}}{R(P,Q)} \right) d\theta(Q) \right] \rho(Q) dL(Q) \tag{16}$$

where, the axisymmetric assumption implies that the field $\phi(P)$ and its derivative are independent of $\theta(P)$ and the differential area element is defined as

$$dS(Q) = \rho(Q)\, d\theta(Q)\, dL(Q) \tag{17}$$

where, *dL(Q)* is the differential length of the generator *L* of the body at a surface point *Q*, where *Q* now is interpreted as an arbitrary point on L only.

The evaluation of the integrands in Equations 15 and 16 requires the evaluation of the following integrals

$$I_1 = \int_0^{2\pi} \left(\frac{e^{-ikR(P,Q)}}{R(P,Q)} \right) d\theta(Q) \tag{18}$$

$$I_2 = \int_0^{2\pi} \frac{\partial}{\partial n}\left(\frac{e^{-ikR(P,Q)}}{R(P,Q)} \right) d\theta(Q) \tag{19}$$

These integrals are singular and the singularities can be removed by using the following regularization scheme.

$$I_1 = IG_1 + IE_1 \tag{20}$$

where,

$$IG_1 = \int_0^{2\pi} \left(\frac{e^{-ikR(P,Q)} - 1}{R(P,Q)} \right) d\theta(Q) \tag{21}$$

and

$$IE_1 = \int_0^{2\pi} \left(\frac{1}{R(P,Q)} \right) d\theta(Q) \tag{22}$$

The integrand in Equation 21 is nonsingular. However, it can be evaluated numerically using a simple Gaussian quadrature formula. The other integrand defined in Equation 22 can be reduced to elliptic integral formulation and evaluated using standard algorithms. Elliptic integral algorithms are available in the numerical toolboxes on most computers [8].

A similar procedure has been used in Reference 2 to evaluate the integrand in Equation 19.

$$I_2 = IG_2 + IE_2 \tag{23}$$

where,

$$IG_2 = \int_0^{2\pi} \frac{\partial}{\partial n} \left(\frac{e^{-ikR(P,Q)} - 1}{R(P,Q)} \right) d\theta(Q) \tag{24}$$

and

$$IE_2 = \int_0^{2\pi} \frac{\partial}{\partial n} \left(\frac{1}{R(P,Q)} \right) d\theta(Q) \tag{25}$$

4. NUMERICAL FORMULATION

Substituting Equations 21 thru' 25 into Equation 10 at different node points i_p and assuming the index of surface elements i_q, the following discretized form of Equation 10, for N nodes on the surface, can be written as

$$\boldsymbol{A}\,\phi = \boldsymbol{B} \tag{26}$$

where, A is an NxN matrix. ϕ and B are N vectors. An example for the hard scatterer where $\frac{\partial \phi}{\partial n} = 0$ is

$$A(i_p, i_q) = \sum_{i_q=1}^{N} I_1 \rho(i_q) dL(i_q) \quad i_p \neq i_q \tag{27}$$

$$A(i_p, i_p) = \sum_{i_q=1}^{N} I_1 \rho(i_q) dL(i_q) - 2\pi \quad i_p = i_q \tag{28}$$

and

$$B(i_p) = -4\pi\phi^i(i_p) \quad \forall i_p = 1..N \tag{29}$$

where, ϕ is an N -dimension vector representing the field strength on the scatterer surface and ϕ^i is the incident field.

5. NON-UNIQUENESS ISSUES

CHIEF method is used to solve the acoustic scattering problem at the natural frequency. Nonuniqueness manifests itself numerically by producing a nearly singular coefficient matrix $\boldsymbol{A}$. At these points, the field strength $\phi(i_p)$ is forced to vanish. The interior integral relations are used as constraints that must be satisfied along with the original formulation. Equation 29 is, then, augmented by additional equations at interior points. These equations differ from Equation 29 in that the additional point doesn't lie on the surface. Thus, no singularity is found and the field strength is zero so the 2π in R.H.S of Equation 29 is dropped.

For axisymmetric bodies, it was found that the axis is a good place to put the interior points [1]. The selection of interior points is based on evaluating the field at some interior points on the axis of symmetry and select only those points where the field doesn't vanish, within a preset accuracy [6]. Augmenting the N discretized equations in Equation 26 by M interior equations results in an overdetermined system of equations which can be solved by the least squares method [1]. The nonuniqueness can be overcome using a simple method based on the CHIEF criterion. The proposed method exploits a non-unique solution at the required characteristic frequency on the scatterer surface. The non-unique solution is, then, used to compute the field at some interior points. The computed field strengths can monitor any nodal points using a simple criterion based on that the point of larger field strength is far from nodal points. The matrix used in the first run is, then, augmented with non-nodal interior points to give the over-determined system of equations which can be solved for the unique surface field.

6. NUMERICAL RESULTS

The numerical example, presented here, is the scattering of a plane incident wave by a rigid sphere and is solved at some characteristic frequencies of the problem. The incoming unit plane wave travels toward the scatterer along the positive direction of z-axis in the cylindrical coordinates described as e^{-ikz}. The surface field ϕ is computed using the proposed method. The results will be verified by comparing with the analytical solution. The benefits of the method will be validated by comparing with the results of CHIEF method with multiple points. On the surface of a hard sphere, the analytical solution of Equation 10 for plane incident wave can be expressed as [8].

$$\phi = \frac{i}{(ka)^2}\sum_{n=0}^{\infty}(-i)^n(2n+1)\frac{P_n(\cos\theta)}{h_n^{(1)'}(ka)} \tag{30}$$

where, ϕ is the total field as defined in (3) and θ is the incidence angle and it has been taken to be *zero* in this application. P_n is the Legender polynomial of order n and h_n is the spherical Hankel function. k is the wave number and a is the radius of the sphere.

Table.1 Field points at different interior points

Normalized interior points z-component	Verified Field
0.90476	0.396
0.80952	0.816
0.71429	1.194
0.61905	1.480
0.52381	1.630
0.42857	1.620
0.33333	1.441
0.23810	1.108
0.14286	0.656
0.04762	0.153
-0.04762	0.433
-0.14286	0.929
-0.23810	1.333
-0.33333	1.602
-0.42857	1.711
-0.52381	1.659
-0.61905	1.460
-0.71429	1.147
-0.80952	0.763
-0.90476	0.351

The performance of each method will be evaluated based on two factors; the computational time and accuracy relative to the analytical solution.

The compared results are taken at the characteristic frequencies; $ka= 4.4934$ which is a fictitious frequency of the normal derivative boundary integral equations [3]. The system of equations is solved numerically using least-squares algorithm which is available in [8].

In applying the proposed method, the computations of the non-overdetrmined system of equations are saved by retaining its matrix into memory after monitoring the proposed CHIEF points. The selected point based on the proposed criterion is used to add, just, one matrix row and then the least-squares method is applied to the over-determined system of equations.

Table. 1 shows the verified field at different interior points to select the appropriate one for CHIEF analysis. The field verification is computed through non-overdetermined system of equation for the integral equation for the characteristic frequency $ka=4.4934$. These points are chosen with constant step on the z-axis of symmetry in cylindrical coordinates. The incident field is propagating along the z-direction. According to the proposed method of selection, the field at the candidate point should exceed a certain threshold. Different trials showed that the threshold $\phi/\phi_i > 0.5$ is sufficient.

According to the above discussion, the most appropriate point, which has the most far field value from zero, is $z=0.524$. Figure 1 shows a comparison between the scattered field distribution on the surface computed without any correction and the analytical solution. Figure 2 shows a comparison between the scattered field distribution on the surface computed using the proposed method, CHIEF with multiple points and the analytical solution.

Figures 3-6 show different comparisons between the analytical solution and different selected CHIEF points according to the above criterion. These comparisons show that the selected points serve to adjust the solution in its neighborhood range, since negative points show closer result to the analytical solution around its neighborhood. Applying the method with two points on both sides of the axes of symmetry shows the most accurate result within these comparisons. Adding, more points don't add more accuracy as shown in Figure 3.

CPU time comparison shows that each additional CHIEF point adds about 5% of the whole process time to be processed. The comparison has been conducted under MATLAB 5.2 on PC-233MHz for 10 surface points and 10 additional interior points.

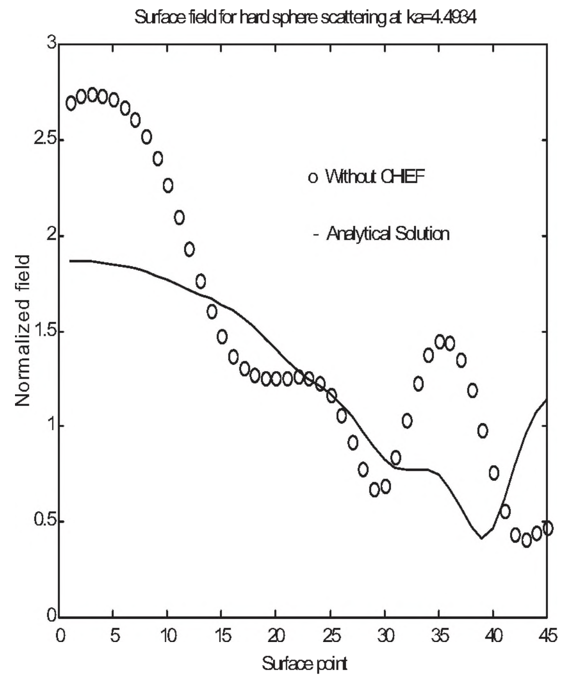

Figure 1. Comparison between the analytical solution at the characteristic frequency *ka=4.4934* and the solution without CHIEF correction.

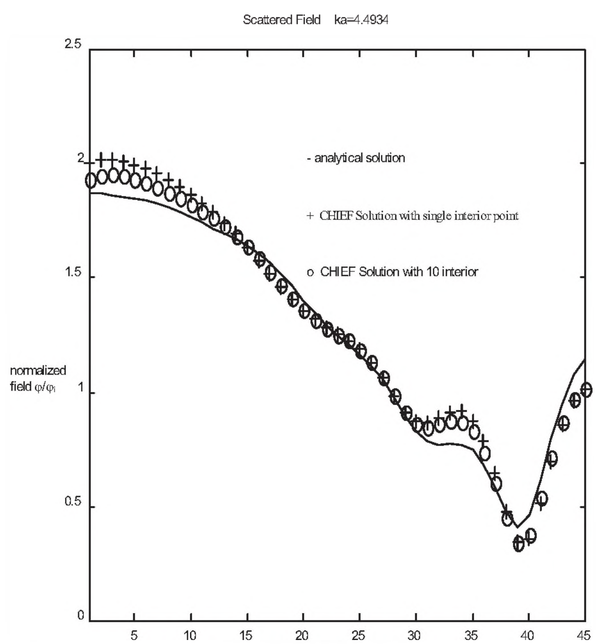

Figure 2. Comparison between different solution methods.

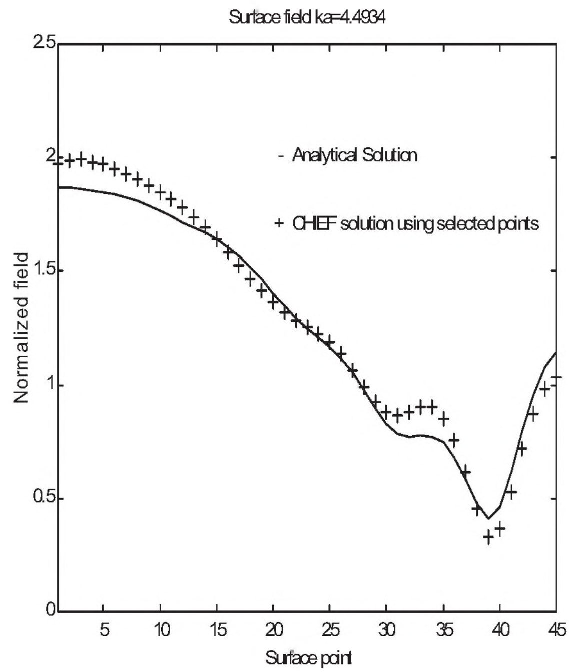

Figure 3. Comparison between analytical solution and a numerical solution at selected point of z=0.52 0.43 and 0.33 respectively.

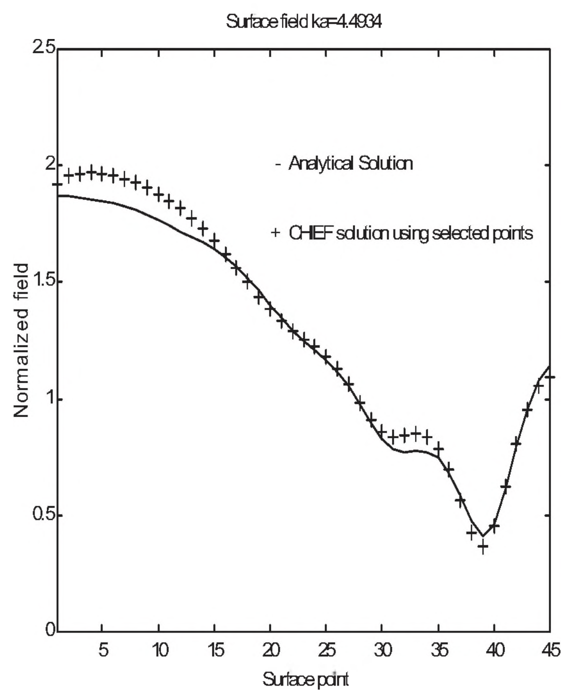

Figure 4. Comparison between analytical solution and a numerical solution at selected point of z=0.62, 0.52, 0.43, -0.62, -0.52 and - 0.43 respectively.

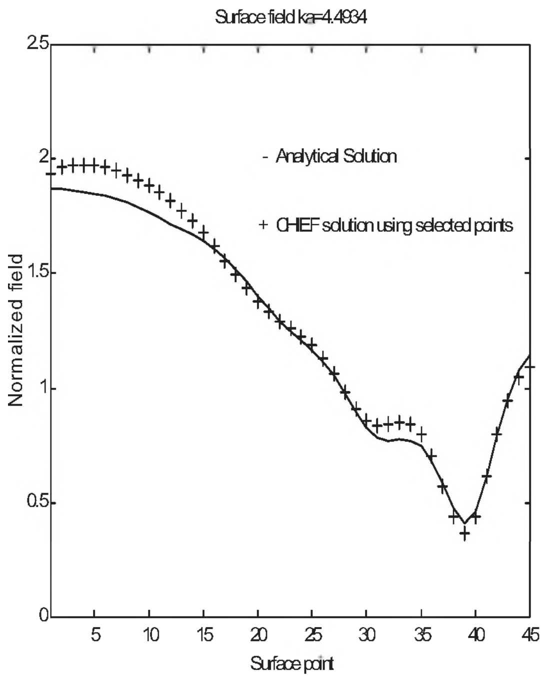

Figure 5. Comparison between analytical solution and a numerical solution at selected point of z= 0.52, 0.43, -0.52 and - 0.43 respectively.

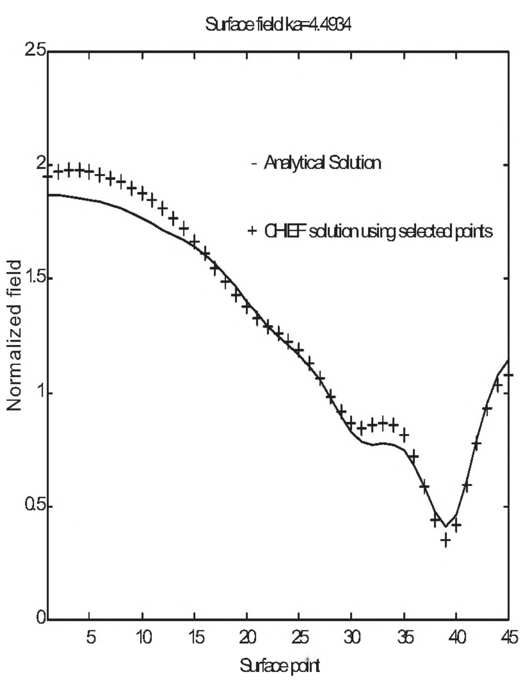

Figure 6. Comparison between analytical solution and a numerical solution at selected point of z= 0.52, 0.43, and -0.52 respectively.

7. Error Analysis

The results obtained in the previous section are then compared based on the error from analytical solution. The error is defined as

$$\textbf{Error} = \phi_{num} - \phi_{ana} \qquad (31)$$

where, ϕ_{num} is the resulted solution form Equation 29 and ϕ_{ana} is the analytical solution given in Equation 30.

Figure 7 shows a comparison between the error trends of the numerical solutions using 10 CHIEF-points and one CHIEF-point at $z = 0.52$ as selected by the discussed criterion in section 5 as an extreme choice. Figure 8 shows another comparison between 10 points solution and the solution of 3-points shown in Figure 6.

The energy error is also compared for different CHIEF-point selections in Table 2. The energy error is defined as the ratio between the energy of the error to the energy of the analytical solution as

$$\text{Energy Error ratio} = \frac{\|\phi_{num} - \phi_{ana}\|}{\|\phi_{ana}\|} \qquad (32)$$

Table.2 Energy Error Ratio for different Chief-point selections.

# Chief Points	10	6	3	1
Energy Error Ratio	0.0448	0.0456	0.0550	0.0592

Table 2 shows the effect of CHIEF-points on the accuracy of the numerical solution. According to these results, the trade-off remains between less accuracy gain as shown in Table 2, and Figures 1thru' 8 and saving computational cost by reducing the solution matrix as discussed in Section 6.

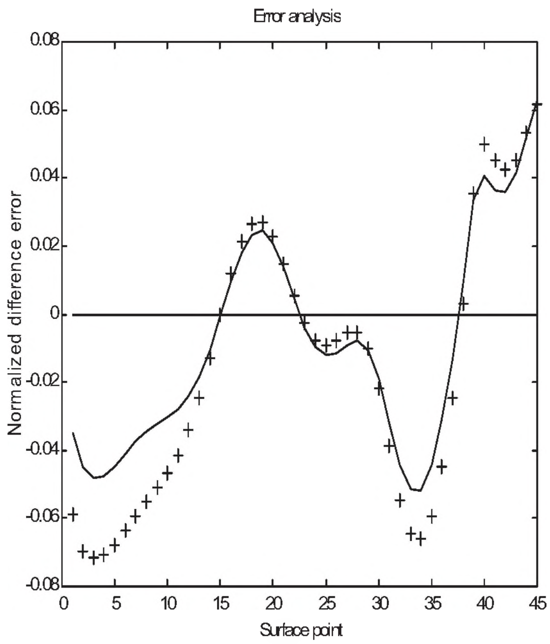

Figure 7. An error comparison between one and 10 CHIEF-point solutions.

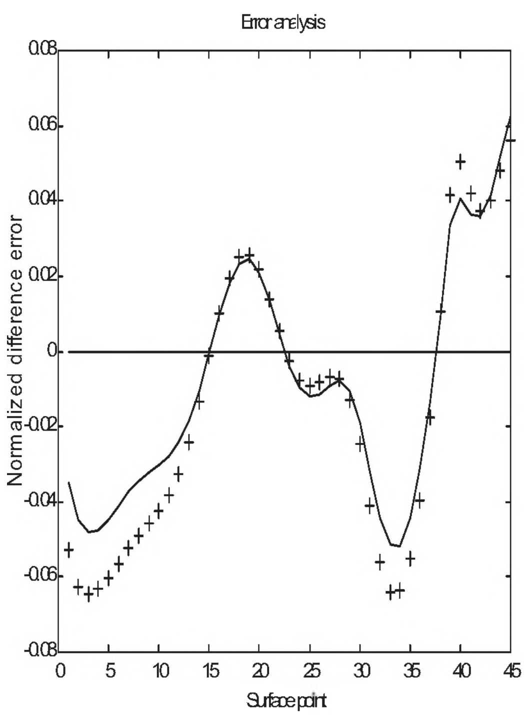

Figure 8. An error comparison between 3 and 10 CHIEF-point solutions.

8. CONCLUSIONS

This work considered the problem of selecting a non-nodal interior point for CHIEF method. This method is essential for solving the acoustic scattering problem at the characteristic frequencies of such problems. The non-uniqueness needed only one interior additional field equation [1]. The selection method, proposed in this work, provided a systematic way which saves both computational time and memory. The results showed convenience with the analytical solution and the traditional solution using multiple interior constraint equations. Error analysis showed that more internal CHIEF-points do not improve accuracy while fewer points reduced the computation time. The proposed method for selecting such points helped in reducing that number. According to the results, the trade-off still remains between less accuracy gain and saving computational cost by reducing the solution matrix.

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