Wavelet-Based Treatment For Nonuniqueness Problem Of Acoustic Scattering Using Integral Equations

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ABSTRACT

Wavelets analysis is a powerful tool to sparsify and consequently speed up the solution of integral equations. The nonuniqueness problem which arises in solving the integral equation of acoustic scattering at characteristic frequencies can be solved at the expense of increasing the problem matrix size. The use of wavelets in expanding the unknown function can efficiently reduce that size since the resulting problem matrix is highly sparse. Examples are discussed for scattering on both acoustically hard and soft spheres. The results are obtained for different Daubechies wavelet orders and sparsification thresholds. A comparison is then presented based on solution accuracy and sparsity ratio.

SOMMAIRE

Les ondelettes que l'analyse est un outil puissant sparsify et accélèrent par conséquent la solution des équations intégrales. Le problème de nonuniqueness qui surgit en résolvant l'équation intégrale de la dispersion acoustique aux fréquences caractéristiques peut être résolu aux dépens d'augmenter la taille de matrice de problème. L'utilisation des ondelettes en augmentant la fonction inconnue peut réduire cette taille efficacement puisque la matrice résultante de problème est fortement clairsemée. Des exemples sont discutés pour la dispersion sur tous les deux acoustique durs et les sphères molles. Les résultats sont obtenus pour différents ordres d'ondelette de Daubechies et seuils de sparsification. Une comparaison est alors présentée basée sur l'exactitude de solution et le rapport d'espacement.

1 INTRODUCTION

The application of boundary integral methods in acoustic scattering suffers from nonuniqueness problem at the characteristic wavenumbers. To overcome this problem, an overdetermined system of equations is derived from the interior Helmholtz integral equation. Although the basic square system of N X N equations has an infinite number of solutions at characteristic wave numbers, only one of these solutions satisfies the interior Helmholtz integral equation. One of the main methods used to overcome nonuniqueness is the addition of the constraints on internal fields at a finite number of points. We can expect therefore that the solution obtained from the overdetermined system will approximate a unique solution [1]. Intuitively, the overdetermined system of equations increases the complexity and thus the extent of the computations [2].

The use of orthogonal wavelets as basis functions can speed up the numerical solution of surface integral equations. The wavelet expansion can adaptively fit itself to various length scales by distributing the localized basis functions over a surface. The cancellation property of the wavelets can eliminate, to a great extent, the coupling between the distant parts of the physical configuration under study. Thus the resultant matrix from moment method is rendered sparse by using wavelet expansion.[3]. The most disrupting fact about the discrete wavelet transform is that the condition number of the matrix changes after transformation [4]. The results, in this work, show that the overdetermined system of equations behaves well after using wavelet expansion. The results are presented for acoustic scattering on both acoustically hard and soft spheres at characteristic wave numbers. The results are obtained for different Daubechies wavelet orders and sparsification thresholds. However, short Daubechies wavelet orders are used since long filters give less sparsity [4]. A comparison is then presented based on solution accuracy and sparsity ratio.

2 ANALYSIS

For a surface S on which a unit normal n, pointing outward, when a harmonic acoustic wave fi impinges upon an acoustically hard body V enclosed by that surface, the resulting integral equation for smooth surface has the following form;

$$4\pi f^{i}(x') + \int_{S} \frac{\partial G(x,x')}{\partial n} f(x')\, dS(x') = 2\pi f(x) \qquad x' \in S \tag{1}$$

A similar equation can also be formulated for acoustically soft sphere by substituting the corresponding boundary condition on pressure

$$4\pi f^i(x') - \int_S \frac{\partial f(x')}{\partial n} G(x,x')dS(x') = 2\pi f(x) \quad x' \in S \tag{2}$$

where G is the free-space Greens function $G(x,x') = e^{-ikR}/R$ where R is the distance between the field point x and a source point x' and k is the wave number and $f()$ is the acoustic field. If we consider only the fully axisymmetric bodies, the scattering Equation (1) and (2) reduce to one-dimensional form.

The unknown field function can be expanded in terms of wavelet base functions. The approximation of $f(x) \in L^2(R)$ at a resolution of 2^{-j}, can be defined as the projection on different wavelet functions

$$f(x) = a_0 + \sum_j \sum_k a_{jk} \, \psi_{jk}(x) \qquad j,k=1..M \tag{3}$$

where a_{jk} is the amplitude of each wavelet at different resolutions (scales) and locations.

Substituting equation (3) in the Galerkin's discretized form of (1) or (2), the integral equation reduces to

$$\mathbf{A}\,\mathbf{W}^T\,\mathbf{X} = \mathbf{B} \tag{4}$$

where **A** is the Galerkin's moment matrix, **X** is the unknown wavelet amplitudes vector, **B** is the incident field vector defined at surface points, and

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \psi_{11}(X_1') & \psi_{11}(X_2') & \ldots & \psi_{11}(X_N') \\ \psi_{12}(X_1') & \psi_{12}(X_2') & . & \psi_{12}(X_N') \\ & . & & \\ \psi_{21}(X_1') & \psi_{21}(X_2') & \ldots & \psi_{21}(X_N') \\ \ldots & \ldots & \ldots & \ldots \\ \psi_{N1}(X_1') & \psi_{N1}(X_2') & \ldots & \psi_{N1}(X_N') \end{bmatrix}$$

The matrix **W** has a squared dimension of $N \wedge N$ and it can be called the wavelet operator. The effective support (nonzero elements) in the operator matrix **W** of a wavelet ψ_{mn} as the interval outside of which the wavelet is practically zero.

For nonuniqueness problem, matrix **A** is then expanded by additional M equations at some selected CHIEF points in the interior of the body V. The selection of CHIEF points are on the axis of symmetry as discussed in [5].

Equation (4) can be rewritten in the form

$$\mathbf{G}\,\mathbf{X} = \mathbf{B} \tag{5}$$

Each element in matrix **G** is, then, set to zero if it does not exceed what can be called sparsification threshold. The resulting sparse matrix allows the use of sparse matrix solvers or multi-level iterations for rapid solution.

Now the solution of equation (5) is the wavelet amplitudes (a^s) defined in equation (3) and consequently the field solution will be the synthesized function

$$\mathbf{F} = \mathbf{W}^{\mathbf{T}}\,\mathbf{X} \tag{6}$$

3 RESULTS

The results are computed for k=4.4934 which is a characteristic wavenumber for both the Helmholtz integral equation and its normal derivative assuming unity characteristic length [6]. The scatterer is taken as a sphere of a unity radius. The incoming unit plane wave travels toward the scatterer along the positive direction of z-axis in the cylindrical coordinates described as e^{-ikz}. The surface field f is computed using equation (5) and (6). The results will be verified via comparison with the analytical solution as in [7]. The benefits of the method will be validated by comparing the accuracy and sparsity ratio for different wavelet basis support lengths and sparsity thresholds.

The accuracy is estimated based on a normalized error with the analytical field. The normalized error (Δ) is defined as the ratio of the error in field strength and the analytical field as follows

$$\Delta = \frac{\| f_{wvl} - f_{ana} \|}{\| f_{ana} \|} \tag{7}$$

where f_{wvl} is the resulted solution form equation (3) and f_{ana} is the analytical solution and $\|.\|$ is the L2 norm.

Another comparison parameter is the percentage sparsity φ which is defined as

$$\varphi = Z_o/Z \times 100\%, \text{ where } \quad Z=Nx(N+M) \tag{8}$$

where Z_o is the number of zeros in the matrix **G** after thresholding, and M is the number of additional interior points (CHIEF points).

Daubechies wavelets are strictly localized in space and approximately localized in spatial frequency [8]. Many recent works employed Daubechies wavelets in solving scattering problems [3], [9], [10], [11] and [12]. So, it has been used in expanding the field function as defined in (3).

The number of points on surface is $N=32$ and the interior points $M=10$ in the case of hard scatterer and $M=8$ in the soft case and all are on the axis of symmetry.

Figures (1) thru' (5) show the scattered field in both cases; acoustically hard and soft spheres, as compared to the corresponding analytical solution at the characteristic wavenumber. The matrix sparsity reaches about its half using Daubechies wavelet of order 2 with a threshold of $5x10^{-3}$ of matrix elements maximum while the solution error does not exceed about 6% as defined in equation (7).

In Figures (2) - thru' (4), the error trend is investigated with respect to the sparsification threshold and sparsity for scattering on acoustically hard sphere. The sparsification threshold range is taken [10^{-4} .. 10^{-2}] since the literatures recommend an average of 10^{-3} [4]. Figures (6) thru' (8), show the error trends for scattering on acoustically soft sphere.

Figures (2) and (3) show steepest increase in the error versus sparsity threshold and accordingly sparsity ratio for scattering on acoustically hard sphere. These results are logical according to the properties of the sparsification process. The results also show that the sparsity does not exceed 60% for an error which is less than 8% for the current method. The wavelet Daubechies of order 2 show better results since it gives error which is less than 6% with higher sparsity ratio. Figure 4 shows rapid deterioration in error with lower sparsity as expected in [4] for longer wavelet (Daubechies of order)3.

Figures (6), (7) and (8) show similar results for scattering on acoustically soft sphere. The Daubechies wavelet of order 2 shows also better results over other lengths of Daubechies family, however, the error shows some fluctuations over the used threshold range in the order of 1%.

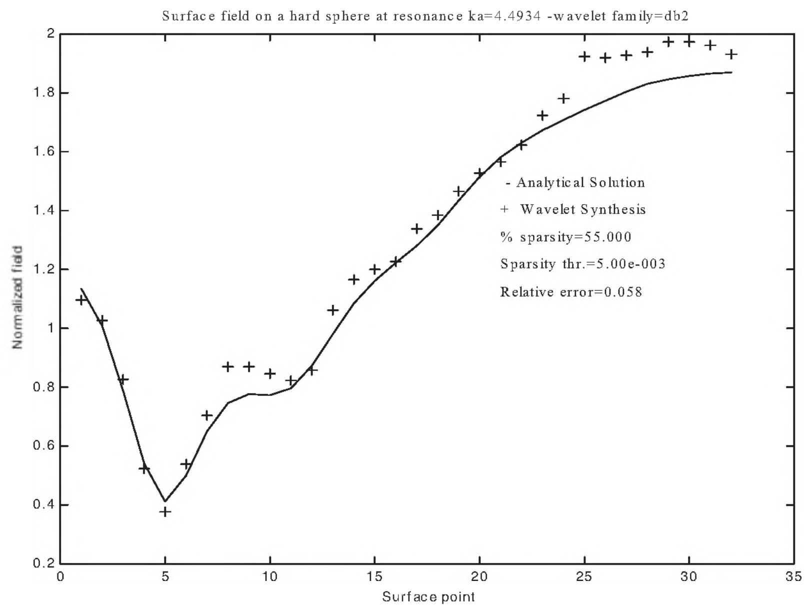

Figure 1 Scattered field of plane wave on an acoustically hard sphere using the Daubechies wavelet of order 2.

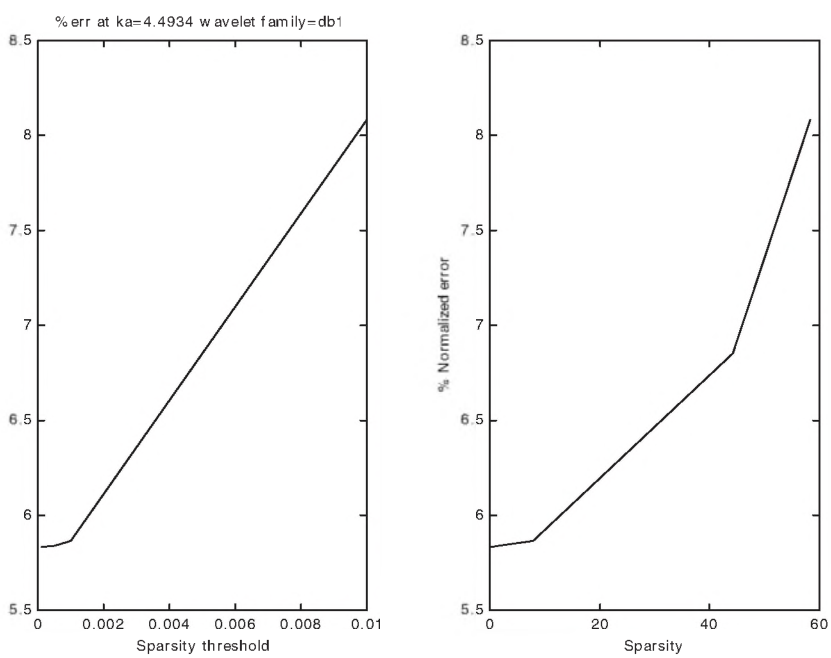

Figure 2. Error trend for hard scattering using Haar wavelet

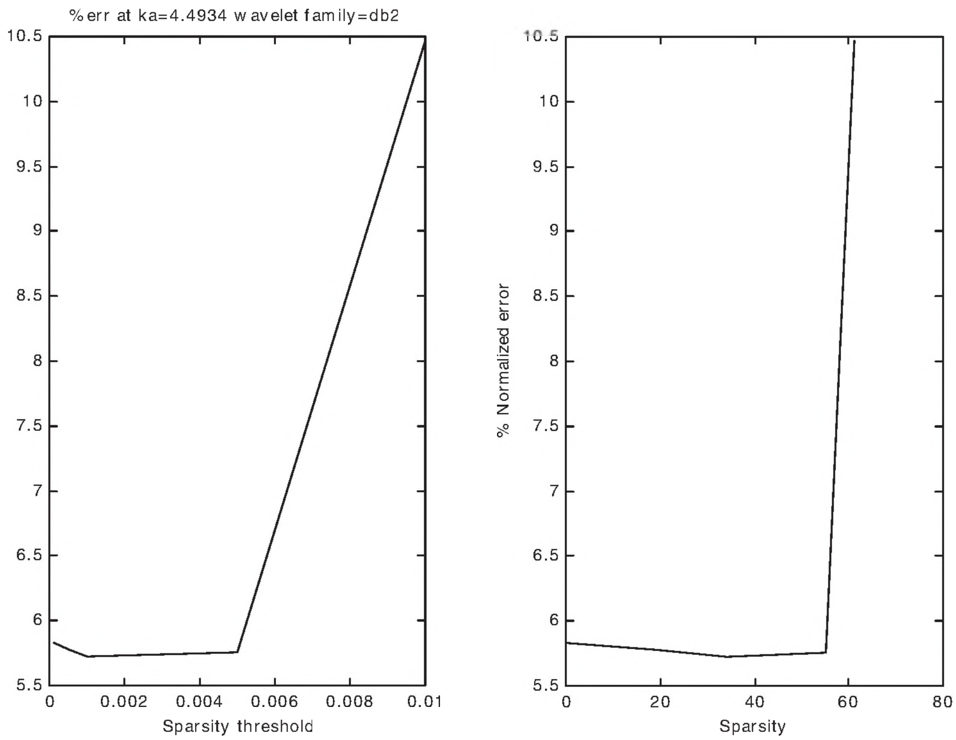

Figure 3 Error trend for hard scattering using Daubechies wavelet of order 2

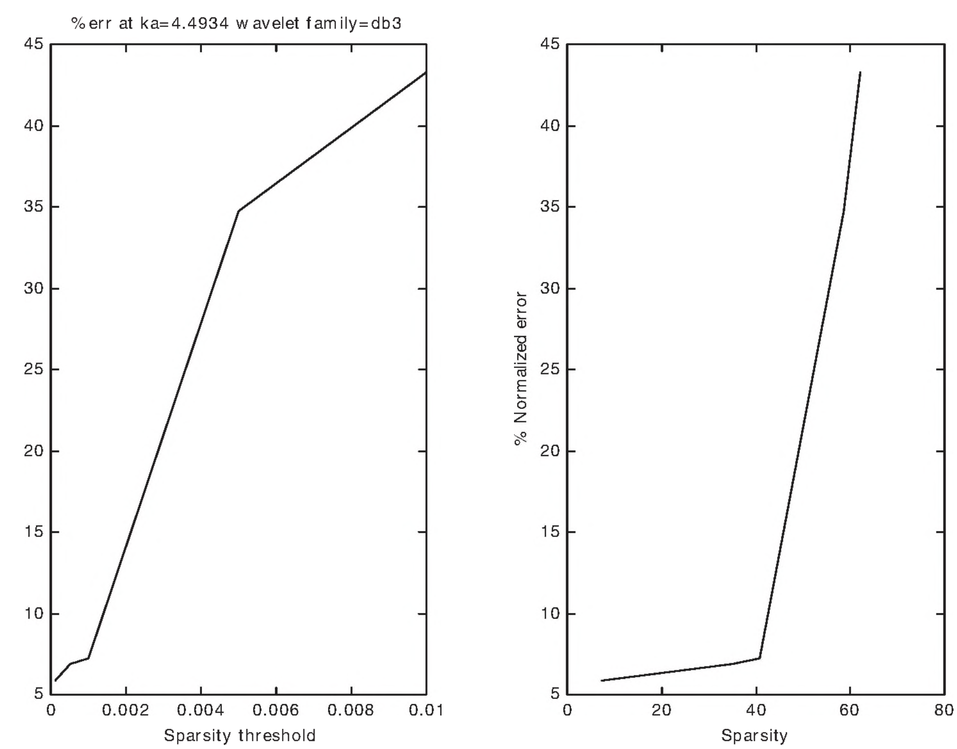

Figure 4. Error trend for hard scattering using Daubechies wavelet of order 3.

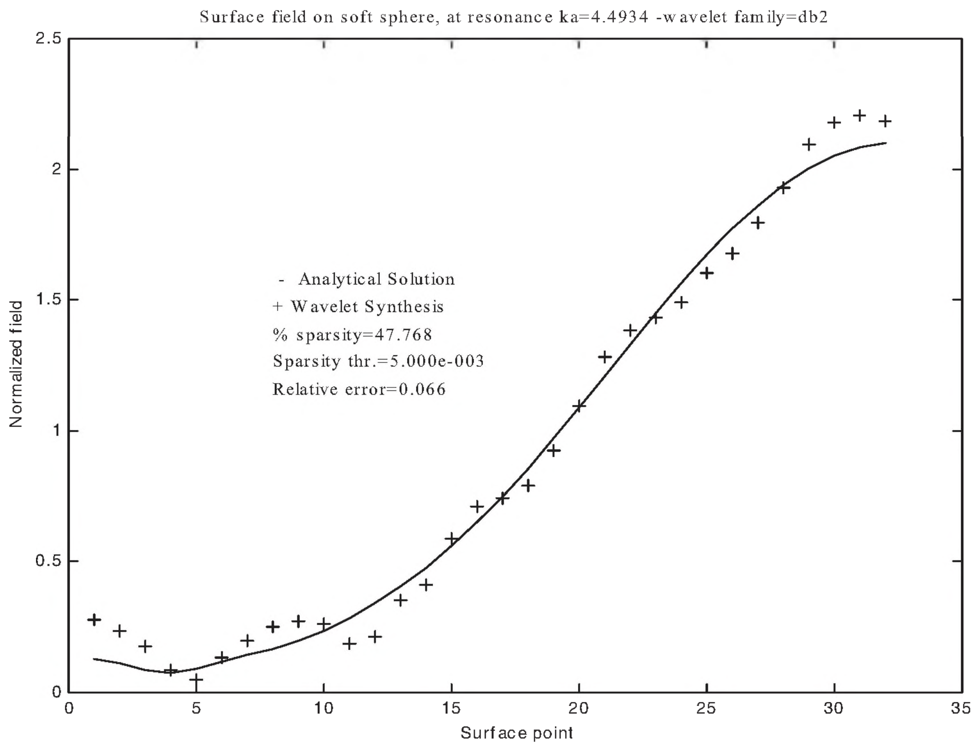

Figure 5 Scattered field of plane wave on an acoustically soft sphere using the Daubechies wavelet of order 2.

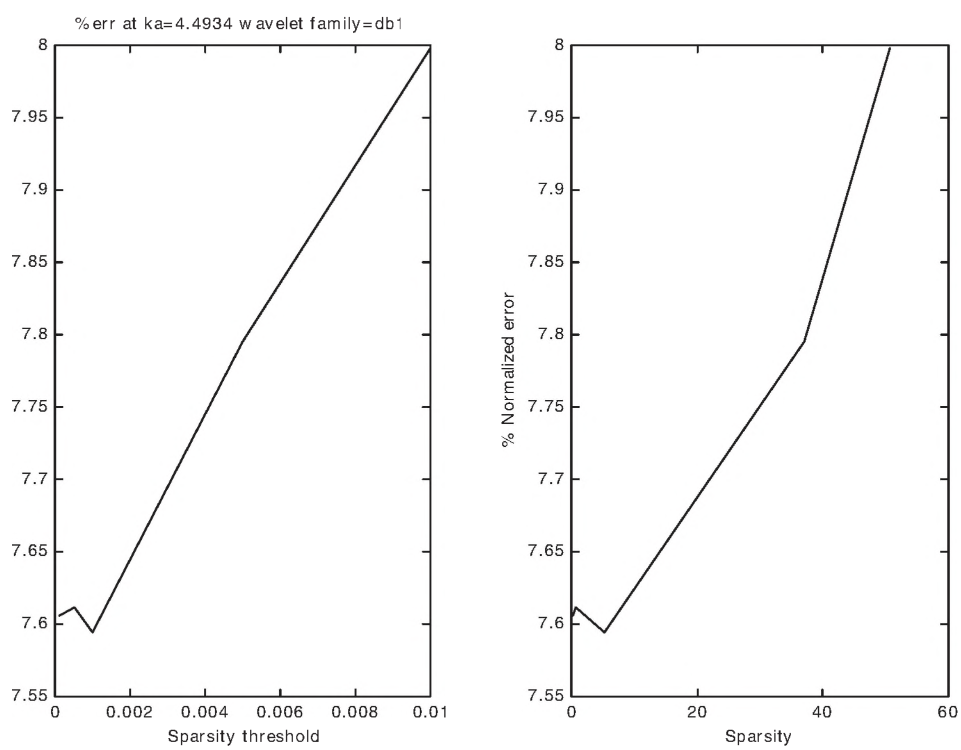

Figure 6. Error trend for soft scattering using Haar wavelet .

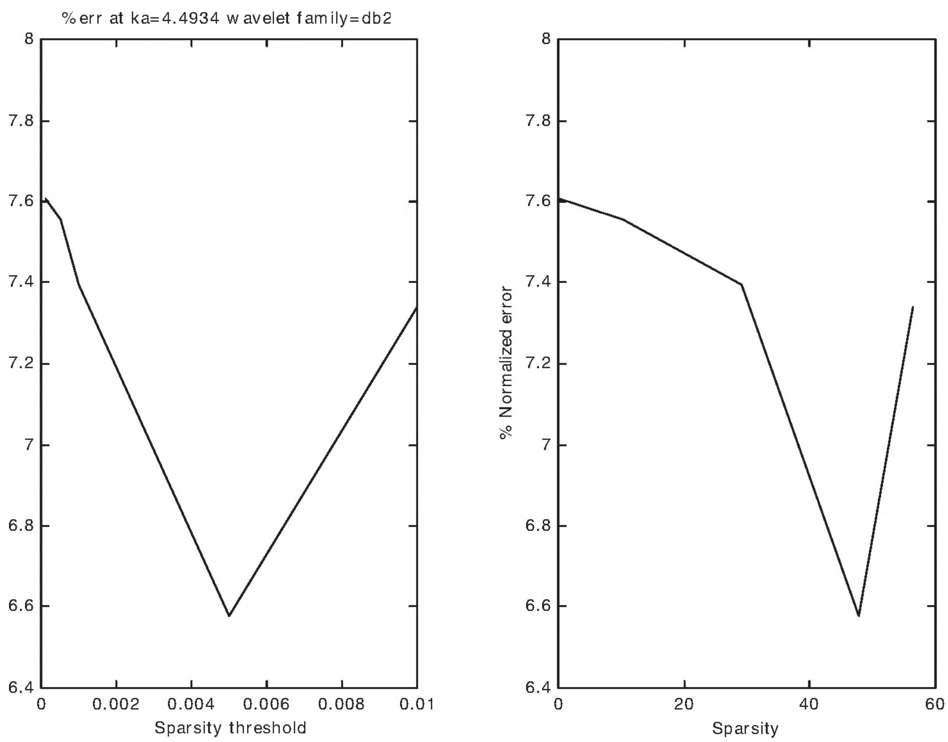

Figure 7. Error trend for soft scattering using Daubechies wavelet of order 2.

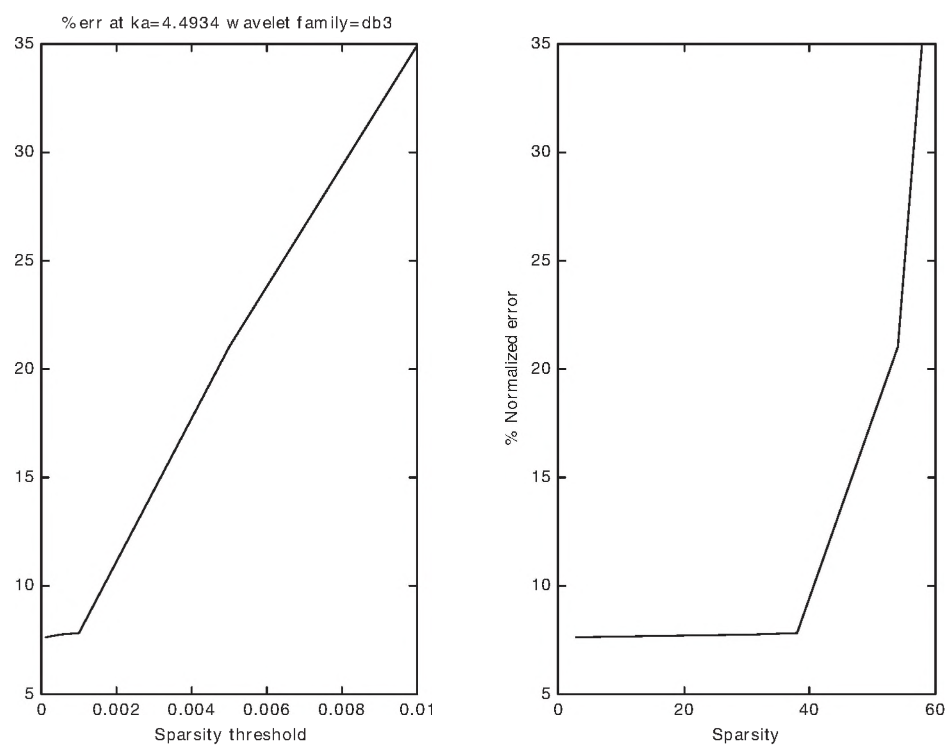

Figure 8. Error trend for soft scattering using Daubechies wavelet of order 3.

4 CONCLUSIONS

The wavelet expansion shows efficient sparsification of resulting matrix in integral equation solution. For acoustic scattering application, the integral equation is solved using wavelet expansion of scattered field for both hard and soft boundary conditions. The nonuniqueness problem, which arises in solving such integral equations at characteristic frequencies, is solved using an overdetermined system of equations of CHIEF method. Different thresholds and wavelet support lengths of Daubechies family are tested for increasing matrix sparsity and the accuracy are computed for each trial. Results show that Daubechies wavelet of order 2 gives better results of lower error with higher sparsity ratio. Haar wavelet does not compete with higher order of 2. Longer length as Daubechies of order 3 shows rapid decrease in the accuracy of the solution with lower sparsity. The sparsification threshold is surveyed in the range of $[10^{-4} - 10^{-2}]$ and the middle of such a range shows better results of higher accuracy and sparsity.

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