IN-PLANE FREE VIBRATION OF CIRCULAR DISKS USING CHARACTERISTIC ORTHOGONAL POLYNOMIALS IN RAYLEIGH-RITZ METHOD

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1. INTRODUCTION

The out-of plane flexural vibrations of the circular disks has been analytically investigated under different boundary conditions using several approximate methods, see for example Rao (1999). In comparison, only a few studies have been carried out on the vibrations within the plane of the disk. However, in many engineering applications involving circular disks, such as railway wheels, grinding wheels, disk brakes etc, in-plane vibrations can play a prominent role causing disk noise and vibration.

Predictions of the in-plane natural frequencies have been treated in a few references. Holland (1966) used trigonometric and Bessel functions to study the free in-plane vibration of circular disks with free edges and presented frequency parameters for different values of Poisson's ratio. Ambati et al (1976) investigated the in-plane vibrations in annular disks with free boundaries. Recently, Farag and Pan (2003) analyzed the modal characteristics of in-plane vibrations of solid disk with clamped outer edge. These studies were carried out for a limited set of boundary conditions. Irie et al. (1984) examined the in-plane vibrations in circular and annular disks using transfer matrix formulation. Natural frequencies were obtained for several radius ratios of annular disks with combinations of free and clamped conditions at the inner and outer edges.

The main objective of the present paper is to provide the natural frequencies of circular disks subject to various combinations of boundary conditions, with relative ease and acceptable accuracy. Most of the studies concerning disk vibrations expressed the mode shapes as a series summation of Bessel functions in the radial direction. In the present analysis, boundary characteristic orthogonal polynomials, first introduced by Bhat (1985) are used as the admissible functions. These functions have some advantageous features such as relative ease of generation and integration, diagonal mass matrix and diagonally dominant stiffness matrix.

The approach presented in this paper is straightforward but more general than the approaches presented previously in the literature. Starting from the constitutive laws and stress-strain relations, the integral expressions for strain and kinetic energies of the disk are presented in polar coordinates. The Rayleigh-Ritz method is employed to solve for the eigenvalues. The radial and circumferential displacement components are expressed in terms of trigonometric functions in the circumferential direction and boundary-characteristic orthogonal polynomials in the radial direction. The frequency parameters are presented for circular disks with different boundary conditions. The accuracy of the eigenvalues is ascertained through comparisons with the existing results from the literature.

2. THEORY

In this section, the in-plane characteristic of a non-rotating circular disk is investigated using the Rayleigh-Ritz method. The material of the disk is assumed to be isotropic with mass density ρ, Young's modulus E and Poisson ratio ν. Let the outer radius of the disk be R, the inner radius be R and the thickness of the disk be h. the radial and circumferential displacement components of a material point on the disk are denoted by u_r, u_θ, respectively.

The expressions for the maximum values of strain and kinetic energies of the disk in polar coordinates are expressed in terms of displacements as:

$$V = \frac{1}{2}\int_0^R\int_0^{2\pi}\frac{E}{1-\nu^2}\left\{\left(\frac{\partial u_r}{\partial r}\right)^2 + 2\nu\left(\frac{u_r}{r}\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}\frac{\partial u_r}{\partial r}\right) + \left(\frac{u_r}{r}\right)^2 + 2\frac{u_r}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{1}{r^2}\left(\frac{\partial u_\theta}{\partial \theta}\right)^2 + \frac{1}{2}(1-\nu)\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right)^2\right\} r\,dr\,d\theta \tag{1}$$

$$T = \frac{1}{2}\int_0^R\int_0^{2\pi}(\dot{u}_r^{\,2}+\dot{u}_\theta^{\,2})\,\rho\, r\,dr\,d\theta \tag{2}$$

The free in-plane vibrational response is assumed to have a sinusoidal variation around the disk. Introducing the non-dimensional parameter $\xi = r/R$, the maximum displacements of the disk may be expressed in the form:

$$u_r(\xi,\theta,t) = \sum_n^{\infty} U_n(\xi)\cos(n\theta)e^{-j\omega t} \quad (3)$$

$$u_\theta(\xi,\theta,t) = \sum_n^{\infty} V_n(\xi)\sin(n\theta)e^{-j\omega t} \quad (4)$$

where n is the circumferential wave number. In this analysis, an orthogonally generated set, based on the work of Bhat (1985), are used to express the radial component of the displacements $U_{n,m} = \sum_m \overline{U}_{n,m}\phi_m(\xi)$ and $V_{n,m} = \sum_m \overline{V}_{n,m}\phi_m(\xi)$. The trail function is chosen to satisfy the boundary conditions at the inner and outer edges of the disk.

As an alternative approach, orthogonal polynomials generated for free conditions can be used as trial functions for the clamped case through the use of artificial springs. Disks are assumed to have elastically restrained boundaries, which require simply adding the energy stored in the springs to the strain energy expressions given in equation (1). The clamped boundary condition is achieved when the springs constants approach infinity. This approach is useful for systems composed of various types of components. For systems in which flexible joints exist between components, the artificial springs are assigned the actual values of stiffness for the joints.

The assumed solutions -equations (3) and (4)- are substituted into the energy equations (1) and (2). Both equations are integrated with respect to θ from $\theta=0$ to $\theta=2\pi$.

Minimizing the natural frequencies with respect to the arbitrary coefficients $\overline{U}_{p,q}$ and $\overline{V}_{p,q}$, results in the following

$$\left([K]-\Omega^2[M]\right)\begin{Bmatrix}\overline{U}_{p,q}\\ \overline{V}_{p,q}\end{Bmatrix} = \{0\} \quad (5)$$

where $\Omega^2 = \rho\omega^2 R^2(1-\nu^2)/E$

The solution of this eigenvalue problem yields the natural frequencies and mode shapes.

3. RESULTS

The energy expressions derived previously are used to obtain the modal characteristics of the disk. The results of the above computations are tabulated for several boundary conditions. The frequency parameters are compared with other results available in the literature to assess the accuracy of the present study. Table 1 presents the dimensionless frequencies for solid disk with free conditions while Table 2 gives the natural frequencies for clamped disks.. The results obtained by the present method are in full agreement with those of other studies, which indicates the accuracy of this method. The frequency parameters are the largest for the clamped-clamped disks, and become smaller in that order for the free-clamped disks, the clamped-free disks and free-free disks. With the increase of the radius ratio, the parameters monotonically increase except for the free-free disks.

Table 1. Frequency parameters for free conditions (numbers in brackets are from Holland (1966)).

(m,n)	1	2	3
1	(1.6176) 1.6175	(3.5291) 3.5289	(4.0474) 4.0472
2	(1.3877) 1.3928	(2.5112) 2.5146	(4.5208) 4.5561
3	(2.1304) 2.1303	(3.4517) 3.4515	(5.3492) 5.349

Table 2. Frequency parameters for clamped conditions using artificial springs (numbers in brackets are from Farag and Pan (2003)).

(m,n)	1	2	3
1	(1.9441) 1.9442	(3.1126) 3.1131	(4.9104) 4.9098
2	(3.0185) 3.0186	(4.0127) 4.0128	(5.7398) 5.7401
3	(3.9116) 3.9118	(4.9489) 4.9491	(6.5537) 6.5542

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