Effect of Boundary Conditions on String Inharmonicity

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ABSTRACT

In studying the vibration of and sound from stringed musical instruments, the string is always considered to be completely fixed. However, this does not reflect the reality, especially for instruments such as guitar and violin where the player presses the strings with fingers. From the results obtained, it is clear that a partially fixed string will produce inharmonicity which will introduce some degree of beat phenomenon. This indicates an important aspect that has not been discussed extensively previously. In this paper, a simple mathematical model has been developed to represent a fixed-partially fixed string. The parameters that cause inharmonicity are discussed and the beat phenomenon between fundamental and inharmonic partials is studied.

1. INTRODUCTION

In classical analysis of string vibrations, the boundary condition has always been set as fixed. However, in many real cases, they are being attached in many ways, which do not always guarantee a completely fixed situation. In this study, the effect of partially fixed string is analyzed which leads to some degree of inharmonicity. This may explain the source of inharmonic partials in some stringed musical instruments such as guitar and violin where the strings are pressed with fingers while playing.

2. BASIC EQUATION

The string is modeled as fixed at one end and partially fixed at the other as shown in Figure 1. The spring stiffness (k) represents the partially fixed nature of the string. When k is infinite, it is equivalent to fixed-fixed string. The wave equation in the string is given by [1]:

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{1}$$

where L is the string length, T is the tension applied, and ρ is the string mass per unit length.

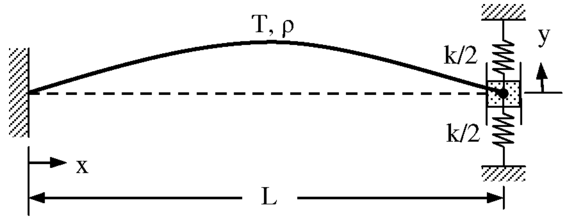

Fig. 1. The string model

The boundary conditions are [2]:

At $x = 0$ $\quad y(0,t) = 0 \quad$ (2a)

At $x = L$ $\quad -ky(L,t) = T\sin\theta \approx T\dfrac{dy}{dx} \quad$ (2b)

Applying the boundary conditions to the wave equation, the frequency equation is obtained as:

$$\tan(\tau L) = -\frac{T\tau}{k} \tag{3}$$

where $\tau = \omega / c$ and $c = \sqrt{T/\rho}$ is the speed of wave traveling. The natural frequencies (ω_n) of the system obtained by solving Eq. (3) as shown in Figure 2, are:

$$\omega_n = c\tau_n \tag{4}$$

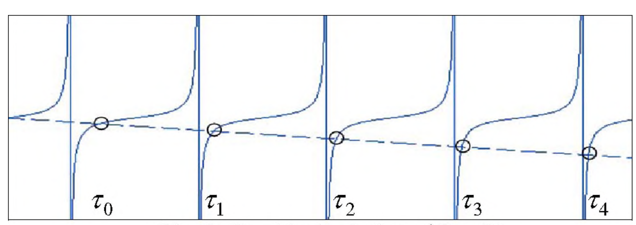

Fig. 2. Graphical solution of Eq. (3)

3. RESULTS

Solving Eq. (3) for various stiffness and string lengths with constant tension, the results are tabulated in

Table 1 where f_0 and f_i are the fundamental frequency and the partials, respectively.

Table 1. Inharmonicity with respect to stiffness (k) and string length (L) with constant tension (T)

Stiffness (k)	Ratio	String length (L)			
		0.25m	0.50m	1.00m	5.00m
5 N/m	f_2 / f_1	2.984	2.974	2.953	2.796
	f_3 / f_1	4.972	4.954	4.906	4.648
	f_4 / f_1	6.959	6.933	6.867	6.500
	f_5 / f_1	8.947	8.913	8.827	8.352
	f_6 / f_1	10.935	10.893	10.788	10.204
50 N/m	f_2 / f_1	2.877	2.778	2.609	2.158
	f_3 / f_1	4.776	4.597	4.289	3.408
	f_4 / f_1	6.680	6.423	5.976	4.697
	f_5 / f_1	8.585	8.253	7.673	6.000
	f_6 / f_1	10.493	10.082	9.371	7.303
500 N/m	f_2 / f_1	2.348	2.169	2.064	2.032
	f_3 / f_1	3.790	3.422	3.179	3.053
	f_4 / f_1	5.257	4.710	4.324	4.063
	f_5 / f_1	6.733	6.009	5.483	5.084
	f_6 / f_1	8.213	7.315	6.652	6.105
5000 N/m	f_2 / f_1	2.005	2.002	1.992	2.010
	f_3 / f_1	3.021	3.006	2.990	3.020
	f_4 / f_1	4.048	4.012	3.988	4.020
	f_5 / f_1	5.087	5.023	4.984	4.949
	f_6 / f_1	6.132	6.035	5.984	6.040
∞ N/m	f_2 / f_1	2.000	2.000	2.000	2.000
	f_3 / f_1	3.000	3.000	3.000	3.000
	f_4 / f_1	4.000	4.000	4.000	4.000
	f_5 / f_1	5.000	5.000	5.000	5.000
	f_6 / f_1	6.000	6.000	6.000	6.000

Tension applied (T) = 100 N

From Table 1, it clearly indicates that the inharmonicity is higher when string length (L) is shorter and stiffness (k) is lower. Also, at low stiffness, the partials are shown to be shifted from their respective harmonic positions to values that are close to odd multiples of the fundamental frequency.

4. DISCUSSION

In classical string vibration analyses, boundary conditions have always been set as fixed which may not correspond to the reality. In current study, the results indicate that the boundary condition has a very significant effect on the behavior of the string by introducing inharmonicity which in turn affects the sound produced. This study finds that the combination of short string with low stiffness in partially fixed string will shift the partials away from their harmonic positions. In the ultimate inharmonic case, the partials are moved to frequencies that are closed to odd multiples of the fundamental value.

5. REFERENCES

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