VERY-LOW-FREQUENCY EVANESCENT LIQUID PRESSURE WAVES

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1. INTRODUCTION

Evanescent liquid sound-pressure waves are standing waves of limited spatial extension. These waves involve variable pressure and liquid-particle velocity but negligible liquid-density variation; they are predicted to occur also in strictly ideal (incompressible, non-viscous) liquids. The free-oscillation frequency reduction observed if resonators (e.g., tuning forks or drinking glasses tapped with spoons) are submerged in water has been shown to be predominantly due to the kinetic energy of the evanescent liquid sound-pressure waves generated by the resonators [e.g., Frosch (2010a, 2010b)]. Such waves are conjectured to occur also, in the cochlear liquid, during spontaneous oto-acoustic emissions [Frosch (2011a, 2011b)]. These spring-driven waves are similar to certain low-frequency evanescent liquid pressure waves, namely to standing water surface waves in cubic vessels. Such gravity-driven waves are discussed, e.g., in Sinick and Lynch (2010) and in Chapter IX of Lamb (1895). The present note is intended to show that classroom demonstrations of these latter waves can provide a plausible introduction to the subject of spring-driven evanescent liquid sound-pressure waves.

2. THEORY OF STANDING LIQUID SURFACE WAVES

2.1. Gravity-driven waves

A possible gravity-driven *travelling* liquid surface wave in an infinitely long wave channel of x-independent width and height is defined by the following equations [e.g., Frosch (2010a)]:

$$\xi(x,z,t) = \zeta_m \cdot \frac{\cosh(kz)}{\sinh(kH)} \cdot \sin(\omega t - kx); \qquad (1)$$

$$\zeta(x,z,t) = \zeta_m \cdot \frac{\sinh(kz)}{\sinh(kH)} \cdot \cos(\omega t - kx). \qquad (2)$$

In Eqs. (1,2), ξ and ζ are the x- and z-components of the displacement of the considered liquid particle from its no-wave place (x,z). A liquid particle can be defined to consist of the water molecules that occupy, at time t = 0, a given cubic millimetre. Since thermal motion and diffusion of the molecules are neglected, in the no-wave state the liquid particle is concluded to keep its initial place and shape. The quantity k in Eqs. (1,2) is the wave number [$k = 2\pi/\lambda$; λ = wavelength]; H is the water depth; at the channel floor, the vertical coordinate z equals zero; $\omega = 2\pi f$ is the angular wave frequency. Eqs. (1,2) are based on the small-displacement approximation, i.e., ζ_m is assumed to be small compared to λ. As shown, e.g., in Frosch (2010a) and in the references quoted there, ω and k are related as follows:

$$\omega^2 = g \cdot k \cdot \tanh(kH). \qquad (3)$$

The quantity $g = 9.81\ \mathrm{m/s^2}$ is the free-fall acceleration. Eqs. (1,2) imply that the involved water particles move on closed elliptical trajectories. At the channel floor (z = 0), the width of these ellipses vanishes.

A *standing* liquid surface wave results if a wave according to Eqs. (1,2) and a similar wave, obtained by the replacement of ωt by (π–ωt), are superimposed:

$$\xi(x,z,t) = 2\zeta_m \cdot \frac{\cosh(kz)}{\sinh(kH)} \cdot \cos(kx) \cdot \sin(\omega t); \qquad (4)$$

$$\zeta(x,z,t) = 2\zeta_m \cdot \frac{\sinh(kz)}{\sinh(kH)} \cdot \sin(kx) \cdot \sin(\omega t). \qquad (5)$$

The liquid particles involved in the standing wave defined by Eqs. (4,5) move back and forth on linear trajectories. The corresponding "sound pressure" p, i.e., the deviation of the liquid pressure from its no-wave value, can be derived from Eqs. (4,5) via the small-displacement version of Newton's second law,

$$\rho \cdot (\partial \vec{v} / \partial t) = -\vec{\nabla} p. \qquad (6)$$

Here, ρ = 1000 kg/m^3 is the water density, and $\vec{v}$ is the liquid particle velocity, having the components $\partial\xi/\partial t$ and $\partial\zeta/\partial t$. Resulting sound-pressure formula:

$$p = a_p \cdot \sin(\omega t);\ \ a_p = a_{p0} \cdot \frac{\cosh(kz)}{\cosh(kH)} \cdot \sin(kx) \cdot \qquad (7)$$

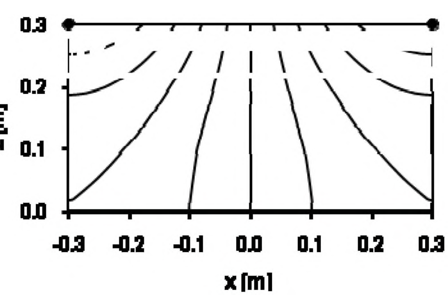

Figure 1. Constant-a_p lines, a_p/a_{p0} = ±0.8, ±0.6, ±0.4, ±0.2, 0.0, according to Eq. (7); a_{p0} is defined in Eq. (8); at the two filled circles, a_p equals ± a_{p0}; H = 0.3 m; $\lambda = 2\pi/k$ = 1.2 m.

In Eq. (7), the quantity

$$a_{p0} = 2\zeta_m \cdot \rho \cdot \omega^2 / [k \cdot \tanh(kH)] = 2\zeta_m \cdot \rho \cdot g \qquad (8)$$

is a pressure constant. Eqs. (4,5) are consistent with time-independent *streamlines* [along which the liquid particles oscillate linearly; e.g., Frosch (2010a)]:

$$\frac{n}{N} = \frac{\sinh(kz)}{\sinh(kH)} \cdot \cos(kx). \qquad (9)$$

In Eq. (9), $n = 1, 2, \ldots, N$ is the running number of the streamlines; see Fig. 2.

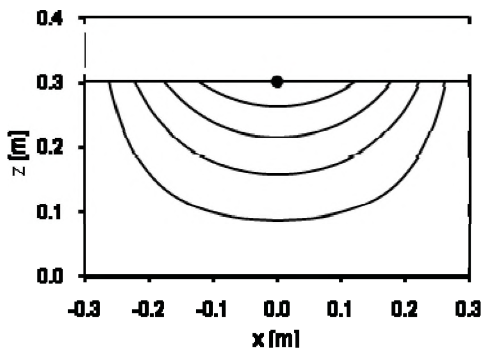

Figure 2. Streamlines according to Eqs. (4,5,9), for $N = 5$, at time $t = 0$.

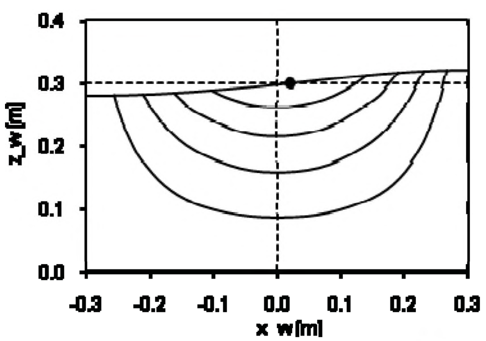

Figure 3. Same as Fig. 2, at time $t = \pi/(2\omega)$; $\zeta_m = 1$ cm.

In Fig. 2 (valid at time $t = 0$), the no-wave places x,z of certain liquid particles are shown, namely of those at the surface and on the streamlines defined by Eq. (9). In Fig. 3 the with-wave places $x_w = x + \xi$, $z_w = z + \zeta$ of the same liquid particles at time $t = T/4$ (where $T = 2\pi/\omega$) are shown.

The water cross section in Fig. 3 is slightly smaller than that in Fig. 2, in contradiction to the assumed incompressibility of the liquid. That inaccuracy occurs because the small-displacement approximation has been used.

2.2. Spring-driven waves

Eqs. (1,2,4,5,6,7,9) are valid also for surface waves in a box model of the cochlear channel with x-independent properties [Frosch (2010a)]. In that case, gravity is negligible; the waves are spring-driven by the basilar-membrane (BM) elements. Eqs, (3,8) must be replaced as follows:

$$\omega^2 = S / \{M + 2\rho / [k \cdot \tanh(kH)]\}; \qquad (10)$$

$$a_{p0} = \zeta_m \cdot S / \{1 + [Mk/(2\rho)] \cdot \tanh(kH)\}; \qquad (11)$$

S is the BM stiffness (spring constant per surface unit, N/m^3), and M is the BM surface mass density (kg/m^2).

Eqs. (10,11) hold for a box model of the cochlear channel with liquid both below and above the basilar membrane.

3. EXPERIMENTS

For a standing wave in a water-filled vessel as shown in Figs. 1-3, the wave length is $\lambda = 1.2$ m, and the corresponding wave number is $k \approx 5.236$ m^{-1}. Eq. (3) yields a frequency of $f \approx 1.09$ s^{-1}, so that this oscillation has indeed a very low frequency and is therefore easy to observe. If one throws visible particles having a density about equal to that of water into a vessel equal or similar to that shown in Figs. 1-3, then one can observe, e.g., that the oscillation amplitude at the floor is much smaller than that at the surface. For $\zeta_m = 1.0$ cm, Eqs. (4,5) yield the following amplitudes:

$\zeta(0.3\text{m}, 0.3\text{m}, T/4) = 2.0$ cm;

$\xi(0.0\text{m}, 0.3\text{m}, T/4) \approx 2.18$ cm;

$\xi(0.0\text{m}, 0.0\text{m}, T/4) \approx 0.87$ cm;

$\xi(0.3\text{m}, 0.0\text{m}, T/4) = \zeta(0.3\text{m}, 0.0\text{m}, T/4) = 0.$

Sinick and Lynch (2010) have described how, with Styrofoam strips, one can excite not only the fundamental oscillation mode (having one node, as shown in our Fig. 3), but also three higher modes (with two, three, or four nodes).

4. CONCLUDING REMARKS

It is justifiable to designate the waves described in Section 2.1 as "evanescent" because of the just mentioned smallness of the oscillation amplitude at the floor, and also because of their similarity to the evanescent waves near submerged resonators. In both cases, standing waves with linear oscillations of liquid particles along time-independent streamlines are generated.

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