CALCULATION OF ENERGY EQUIVALENT NOISE LEVEL FROM TRUNCATED STATISTICAL NOISE LEVEL DISTRIBUTIONS

by

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ABSTRACT

Formulas for calculating the ensemble energy equivalent noise level for a population of sources are given for truncated uniform, normal, and Gamma distributions.

When assessing the effectiveness of a regulation limiting the noise emission of a certain product (say, trucks, for illustration), the following procedure is often followed. A statistical distribution of "current" (i.e., without regulation) noise levels is obtained (e.g., Figure 1(a)) for the population of that product. Its energy equivalent ensemble noise level is calculated and used in a noise impact computation. Then, assumptions are made on how the regulation will change the noise level distribution, and a new equivalent noise level is obtained and fed in turn into a noise impact model. The difference in noise impact may be regarded as the effect of the regulation.

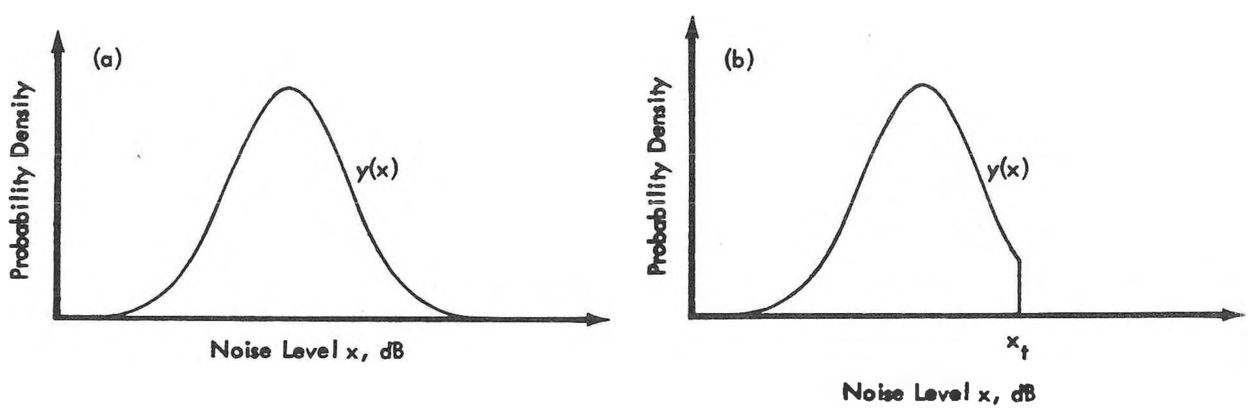

Figure 1. Statistical Distributions of Noise Levels: (a) Full, (b) Truncated

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The simplest assumption about the effect of a noise level regulation on a statistical distribution of noise levels from a source population is that the upper tail is truncated from the distribution at a known noise level. It is assumed that the parameters of these distributions have been estimated and are known quantities. For the normal distribution, these estimations are well-known. For the Gamma distribution, estimation procedures may be found for instance in Bury 1975, page 311. Only uniform, normal and Gamma distributions are treated here since no other closed form solutions could be found which are readily computed with a programmable pocket calculator.

Denoting noise levels by the symbol x with units dB, and probability density by the symbol y, the energy equivalent ensemble noise level L_{ep} (in dB) is defined by

$$L_{ep} = 10 \log \left\{ \int_{-\infty}^{\infty} y(x)\, z^{x}\, dx \right\} \tag{1}$$

where z is a constant equal to $10^{0.1}$. The area under y(x) must, of course, equal unity for y to be a proper probability density function. The subscript "ep" is chosen to distinguish this statistical energy equivalent level of a noise source population, from the energy equivalent level computed over a time interval of a fluctuating noise level signal, usually denoted by L_{eq}.

(a) Uniform Distribution

The formulas for uniformly distributed noise levels are as follows:

$$y(x) = f_u(x; a, b) = \begin{cases} 0 & , \quad x < a \\ \dfrac{1}{b - a} & , \quad a \leq x \leq b \\ 0 & , \quad b < x \end{cases}$$

$$L_{ep} = 10 \log \frac{z^b - z^a}{(b - a)\, \ln z} \tag{2}$$

When the upper portion is truncated above x_t, b is simply replaced by x_t. Figure 2 shows equation (2) in dimensionless form.

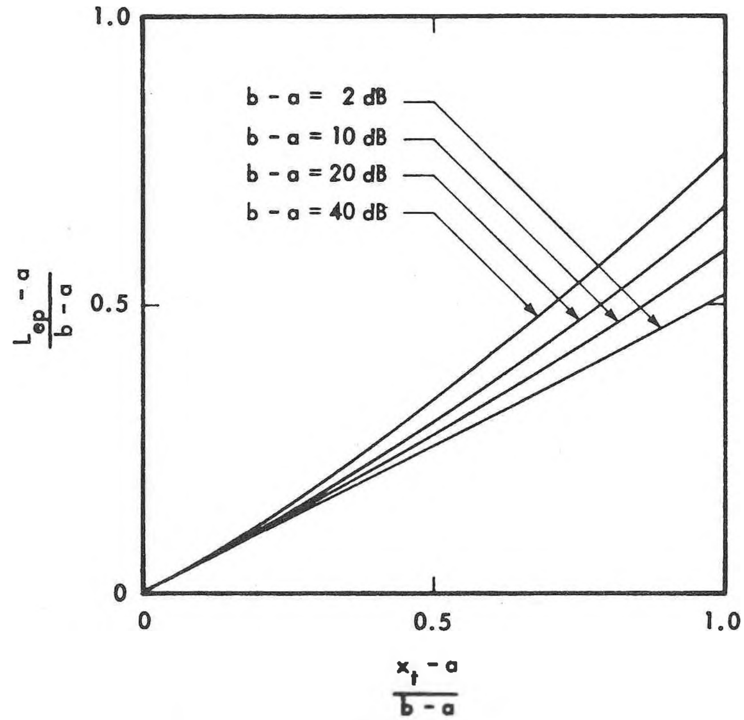

Figure 2. Dimensionless L_{ep} as a Function of Dimensionless Truncation Value (Uniform Distribution)

(b) Normal Distribution

The full normal probability density is

$$y(x) = f_N(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad (3)$$

where μ is the mean, and σ the standard deviation. The cumulative distribution function F_N is

$$F_N(x; \mu, \sigma) = \int_{-\infty}^{x} f_N(t; \mu, \sigma)\, dt \quad (4)$$

If the upper tail of f_N is truncated at x_t (Figure 1(b)), y(x) must be corrected such that the area under y remains equal to unity:

$$y(x) = f_{N,t}(x;\mu,\sigma,x_t) = \frac{1}{\sigma\sqrt{2\pi}\, F_N(x_t;\mu,\sigma)} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \qquad (5)$$

$$\text{for } -\infty < x < x_t\,, \qquad \text{zero elsewhere.}$$

Substituting equation (5) into equation (1) yields the following result after some algebra:

$$L_{ep} = \mu + \sigma^2 \frac{\ln z}{2} + 10 \log \left\{ \frac{F_N(x_t;\mu + \sigma^2 \ln z, \sigma)}{F_N(x_t;\mu,\sigma)} \right\} \qquad (6)$$

The second term is the familiar $0.115\,\sigma^2$ term in the expression for the energy equivalent level for a normal distribution of sound levels. The third term may be called a "truncation correction" which reduces to zero for $x_t \to \infty$ as should be expected.

The cumulative normal distribution F_N is tabulated in any textbook on statistics. Simple pocket calculator programs also exist to evaluate F_N.

Figure 3 shows equation (6) in dimensionless form. This graph can be used to evaluate equation (6), interpolating between curves when necessary.

(c) <u>Gamma Distribution</u>

The full, three-parameter Gamma probability density function is (Figure 4):

$$y(x) = f_G(x;\beta,\alpha,\lambda) = \frac{1}{\alpha\Gamma(\lambda)} \left(\frac{x-\beta}{\alpha}\right)^{\lambda-1} \exp\left(-\frac{x-\beta}{\alpha}\right) \qquad (7)$$

$$\text{for } x > \beta\,, \qquad \text{zero elsewhere,}$$

where β is the location parameter, α the scale parameter, and λ the shape parameter. $\Gamma(\lambda)$ is the complete Gamma function defined by

$$\Gamma(\lambda) = \int_0^\infty t^{\lambda-1} e^{-t}\, dt$$

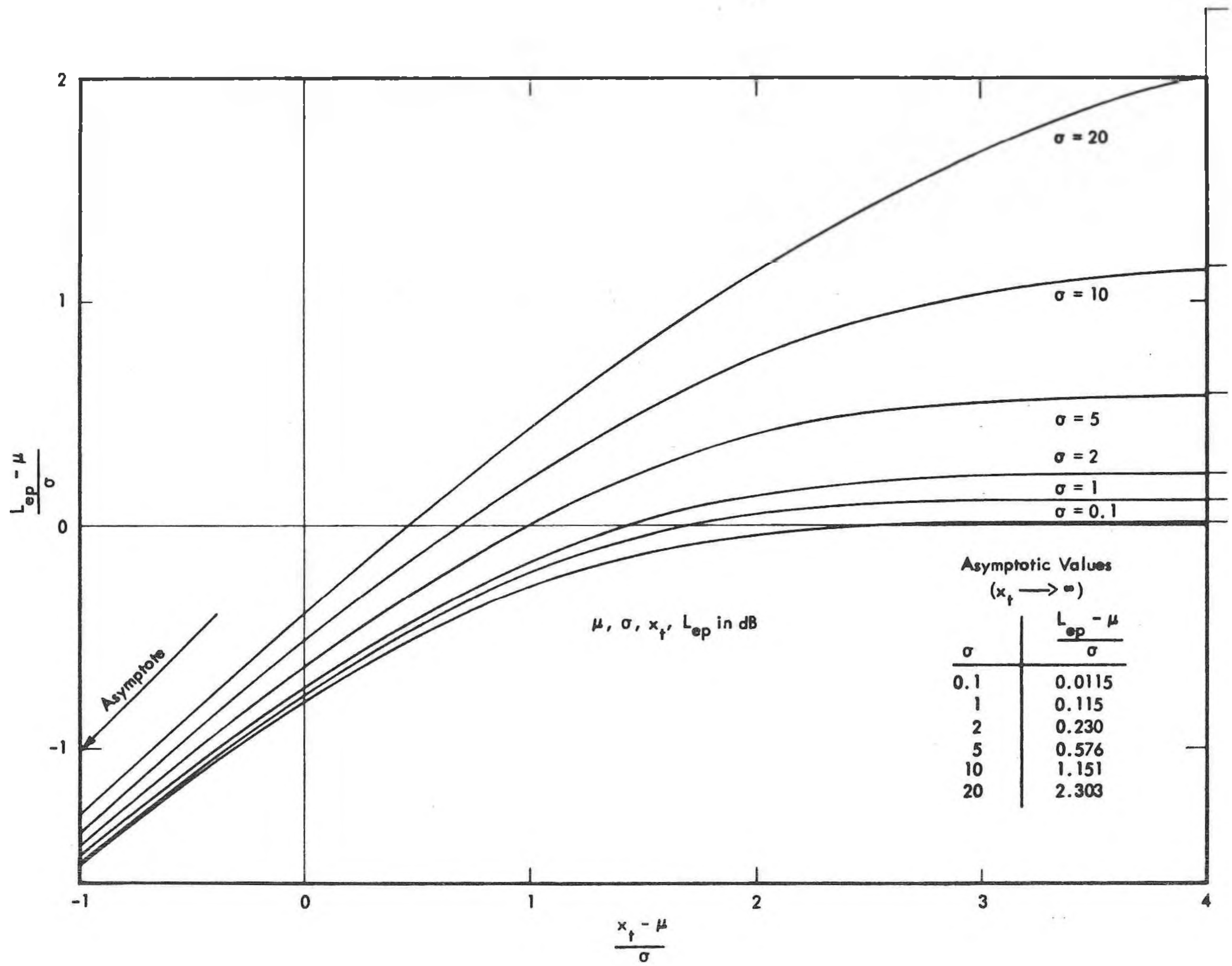

Figure 3. Dimensionless L_{ep} as a Function of Dimensionless Truncation Value (Normal Distribution). Tick Marks Outside Right Margin Indicate Asymptotes.

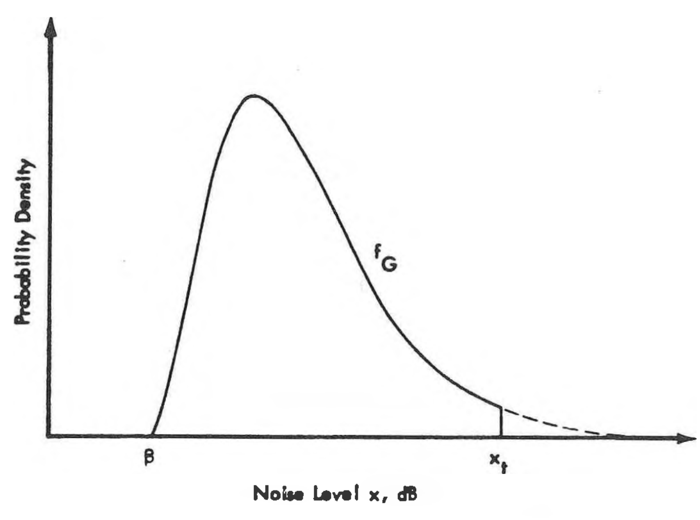

Figure 4. Gamma Distribution

Note that the arithmetic mean of the three-parameter Gamma distribution is $\beta + \alpha\lambda$, and the variance is $\alpha^2\lambda$.

The cumulative distribution function F_G is

$$F_G(x;\beta,\alpha,\lambda) = \frac{1}{\Gamma(\lambda)} \int_{\beta}^{x} \left(\frac{x-\beta}{\alpha}\right)^{\lambda-1} \exp\left(-\frac{x-\beta}{\alpha}\right) \frac{dx}{\alpha} \tag{8}$$

which can be written

$$F_G(x;\beta,\alpha,\lambda) = \frac{\gamma_\lambda\left(\frac{x-\beta}{\alpha}\right)}{\Gamma(\lambda)} \tag{9}$$

where γ_λ is the "incomplete" Gamma function defined by

$$\gamma_\lambda(x) = \int_0^x t^{\lambda-1} e^{-t} dt \tag{10}$$

This leads to the Gamma probability density with truncated upper tail at x_t:

$$y(x) = f_{G,t}(x;\beta,\alpha,\lambda,x_t) = \frac{1}{\alpha\gamma_\lambda\left(\frac{x_t-\beta}{\alpha}\right)} \left(\frac{x-\beta}{\alpha}\right)^{\lambda-1} \exp\left(-\frac{x-\beta}{\alpha}\right) \tag{11}$$

for $\beta \le x \le x_t$, zero elsewhere.

Inserting equation (11) into equation (1) yields the following result after some algebra:

$$L_{ep} = \beta - 10\lambda \log(1-\alpha\,\ell n\, z) + 10\log\left\{\frac{\gamma_\lambda\left[\frac{x_t-\beta}{\alpha}(1-\alpha\,\ell n\, z)\right]}{\gamma_\lambda\left(\frac{x_t-\beta}{\alpha}\right)}\right\} \tag{12}$$

The third term is again the truncation correction term as similarly observed with the normal distribution (equation (6)) reducing to zero for $x_t \to \infty$. With one exception, L_{ep} should always exist for finite x_t. The exception is that the calculation method breaks down when values of the scale parameter $\alpha \geq 4.343$ (= $1/\ell nz$), which is not considered a serious restriction. Values of α will usually range between 0.5 and 2.5 when operating on environmental noise data.

Appendix A gives an algorithm for calculating the incomplete Gamma function on a programmable pocket calculator or a computer.

Figure 5 shows curves representing equation (12) in dimensionless form using the Gamma standardized variable

$$\frac{\frac{x}{\alpha} - \lambda}{\sqrt{\lambda}}$$

One may note the basic similarity of Figure 5 with Figure 3. However, since the Gamma distribution has a finite lower tail, the curves stay finite on the left side.

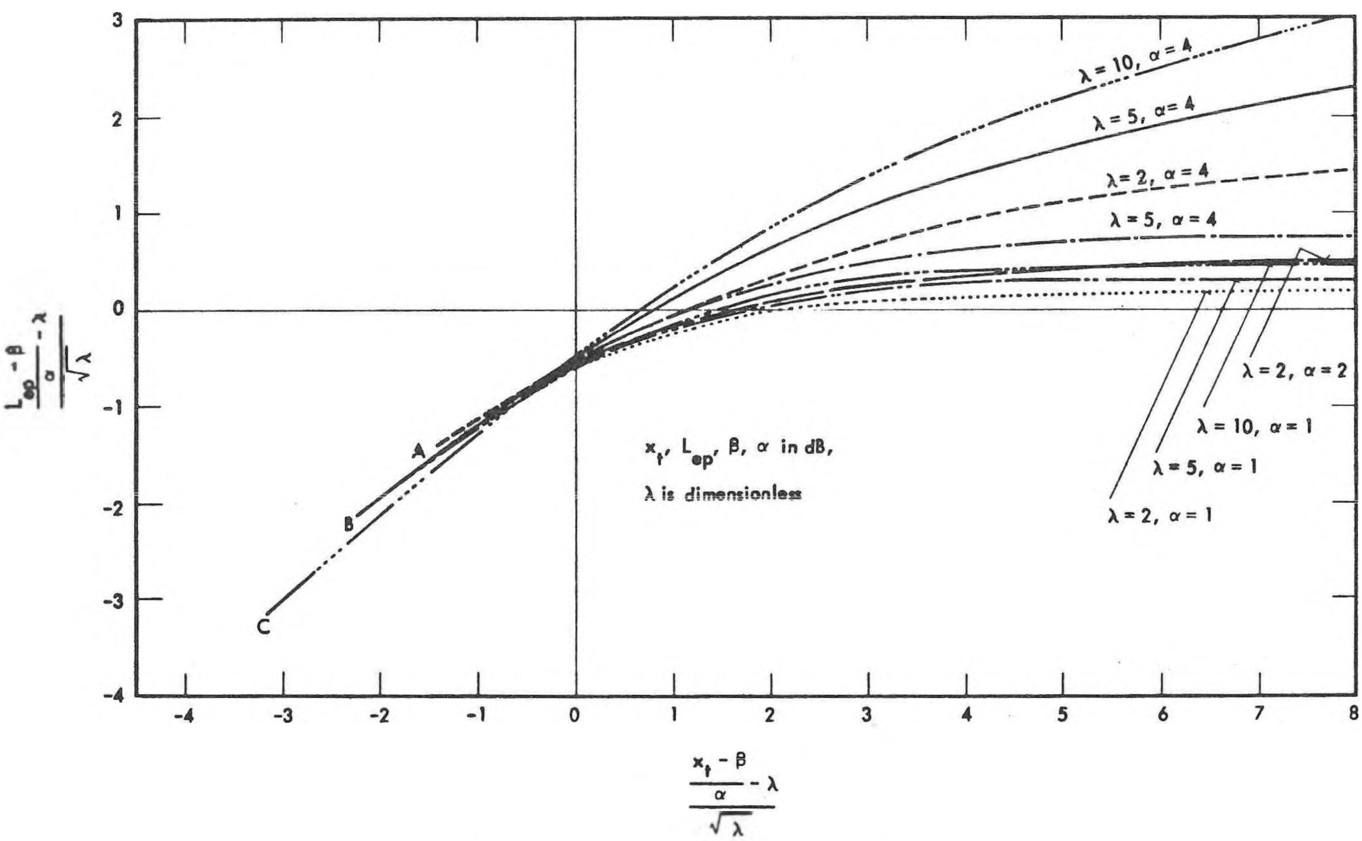

Figure 5. Dimensionless (Reduced) L_{ep} as a Function of Dimensionless (Reduced) Truncation Value (Gamma Distribution). A, B, C are Endpoints of Curves with Same Shape Parameter λ (A: $\lambda = 2$, B: $\lambda = 5$, C: $\lambda = 10$.)

The effectiveness of a regulation is best assessed by estimating its effect on a population of noise sources as a function of time. This has been examined in Plotkin 1974, Plotkin and Sharp 1974, and Plotkin 1977. These references either assume normal models, or they use measured noise level distributions without fitting them with a mathematical model which is a more accurate and also more time-consuming technique than the one discussed in this paper. The formulations developed in this paper are used in Wyle Research 1979 (see references), a document developed for use by local governments, whereas the previously mentioned references stress the use by a federal government.

References

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APPENDIX A

ALGORITHM FOR INCOMPLETE GAMMA FUNCTION EVALUATION

Definition of incomplete Gamma function:

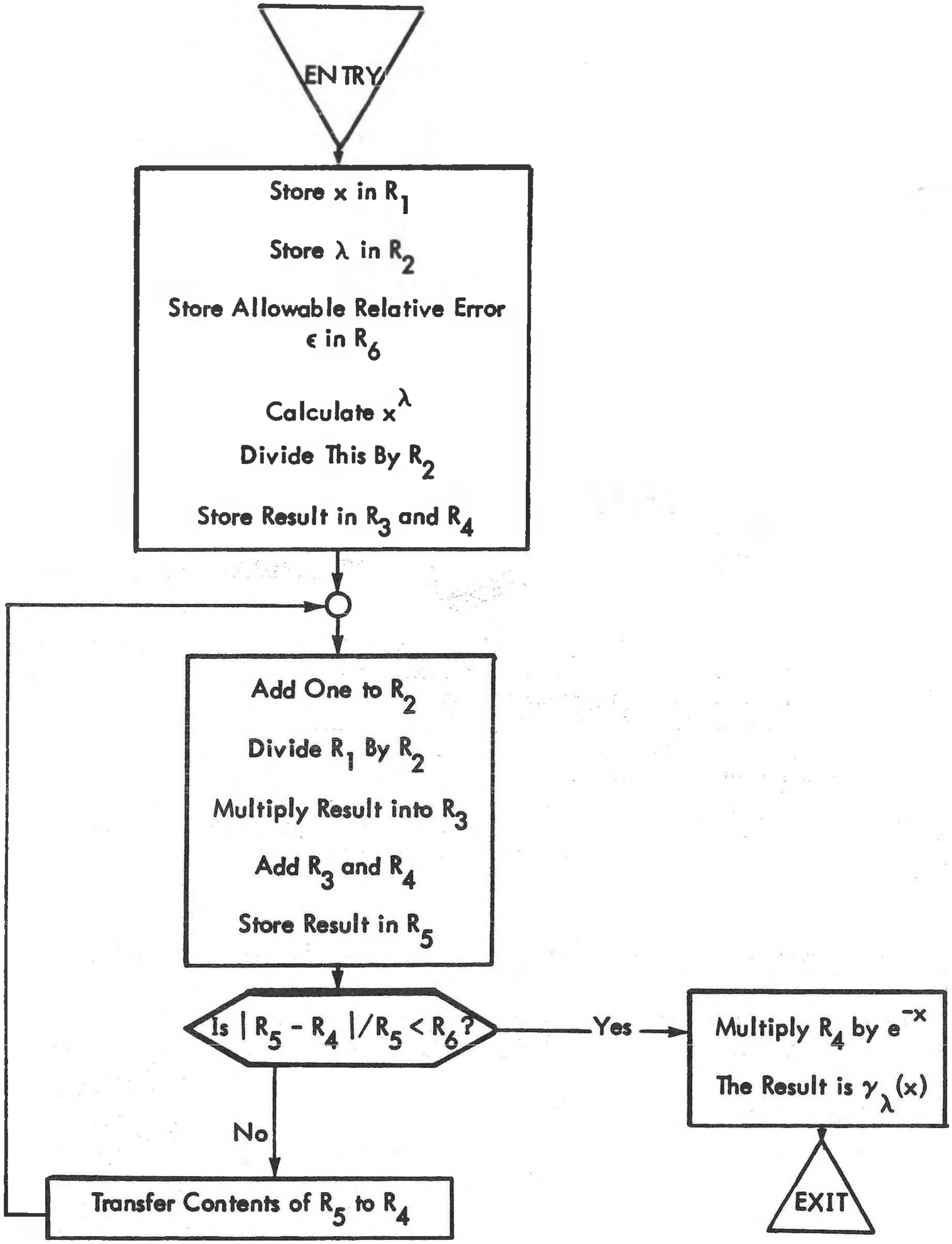
$$\gamma_\lambda(x) = \int_0^x t^{\lambda - 1} e^{-t} \, dt \tag{A1}$$

where $\lambda > 0$, $x > 0$. This is also equal to:

$$\gamma_\lambda(x) = x^\lambda e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{\lambda(\lambda + 1) \ldots (\lambda + n)} \tag{A2}$$

Algorithm for evaluating equation (A2):

$R_1, \ldots . R_6$ are storage registers.



Examples:

$\gamma_1(2) = 0.86466$

$\gamma_1(0.1) = 0.09516$

$\gamma_2(10) = 0.99950$

$\gamma_{10}(5) = 11549.8$