Some Simple Formulae for Normal Mode Wave Numbers, Cutoff Frequencies, and the Number of Modes Trapped by a Sound Channel*

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ABSTRACT

To a good first approximation acoustic propagation in an underwater sound channel is dominated by a finite number of trapped modes. However, exact solutions are known for only a few special cases, making it necessary in general to use numerical methods to solve the normal mode equation. But often one is interested only in the gross features, such as the number of modes or cutoff frequencies, and one does not need the detail provided by a complete normal mode calculation. Even if a normal mode calculation is desired, the computation time can be reduced considerably if the mode wavenumbers can be estimated in advance. In such a case, the WKB method can be used to obtain formulae which, although they are approximate, are given in closed form. In this paper formulae based on exact and WKB solutions are presented for the number of modes trapped in some simple sound channels and for the wave numbers and cutoff frequencies associated with these modes. The number of trapped modes is shown to depend on the gross features of the sound channel, while the distribution of modal wave numbers depends to a greater degree on the details of the sound speed profile shape.

RESUME

Dans une bonne première approximation, la propagation acoustique dans un canal de son sous-marin est dominée par un nombre fini de modes piégés. Cepandant, des solutions exactes ne sont connues que pour quelques cas spéciaux, ce qui oblige en général à utiliser des méthodes numériques pour résoudre l'équation du mode normal. Souvent pourtant, le chercheur ne s'intéresse qu'aux caractéristiques brutes comme le nombre de modes ou les fréquences de coupure et il n'a pas besoin de la quantité de détails fournie par un calcul complet du mode normal. Même, lorsqu'un calcu du mode normal est voulu, le temps de calcul peut être considérablement réduit si les nombres d'onde du mode peuvent être estimés auparavant. Dans un tel cas, la méthode WKB peut servir à obtenir des formules qui, bien qu'elles soient approximatives, se présentent sous une forme fermée. Dans cette communication, l'auteur présente des formules basées sur des solutions exactes et sur des approximations WKB pour le nombre de modes piégés dans des canaux de son simple et pour les nombres d'onde et les fréquences de coupure liés à ces modes. Il démontre que le nombre de modes piégés depend des caractéristiques brutes du canal de son, tandis que la distribution des nombres d'onde modaux dépend dans une plus grande mesure des détails de la forme du profil de la vitesse du son.

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1 Introduction

Acoustic propagation in an underwater sound channel is dominated by a finite number of trapped modes whose wavenumbers depend on the sound speed profile in the channel. Exact solutions are known for only a few special profiles, making it necessary in general to use numerical methods to solve the normal mode equation. But often one is interested only in the gross features, such as the number of modes or the cutoff frequencies, and one does not need the detail provided by a complete normal mode calculation. Moreover, even if a normal mode calculation is desired, the computation time can be reduced considerably if the mode numbers can be estimated in advance. In such cases the WKB (after Wentzel-Kramers-Brillouin and others) method can be used to obtain formulae which, although they are approximate[1], are given in closed form.

In this paper, formulae based on WKB solutions are presented for the number of modes trapped in some simple sound channels and for the wave numbers and cutoff frequencies associated with these modes. The number of trapped modes is shown to depend on the gross features of the sound channel, while the distribution of mode wavenumbers depends to a greater degree on the details of the profile shape. Results are presented for the square (isovelocity-channel) profile, the parabolic profile, and the bilinear profile. An example shows how the simple formulae can be applied to a realistic ocean environment.

While the analysis is presented in terms of underwater acoustics, the results are applicable to other areas, such as transmission in an inhomogeneous waveguide or to the solution of the Schrodinger equation. Most of the results presented here have been known for some time[2], but what is new is that the formulae for some of the more complicated waveguides can be put in the same functional form as the well known formulae for the ideal waveguide. The physical parameters, such as frequency, depth, and sound speed are easily distinguished from the details of the shape of the sound speed profile, which can be treated as a dimensionless quantity. The very simple form of the expressions makes them useful for back of the envelope calculations or for use with a pocket calculator. Moreover, using the same functional form for the expressions allows the effect of the shape of the sound speed profile to be easily seen.

2 The normal mode equation

The normal mode equation can be written as,

$$u_n''(z) + [\omega^2/c^2(z) - k_n^2]u_n(z) = 0 \qquad (1)$$

where,

z is the depth coordinate (increasing with depth from the surface),

$\omega = 2\pi f$ is the angular frequency,

$c(z)$ is the sound speed as a function of depth,

k_n is the wave number or eigenvalue,

$u_n(z)$ is the normal mode function,

and u'' denotes the second derivative of u with respect to z.

In general one wants to determine the normal mode wave numbers k_n and the associated mode functions $u_n(z)$, subject to certain boundary conditions. Notice the quantity $[\omega^2/c^2(z)-k_n^2]$ which will be important in the discussion later; in particular, it is equal to zero at a turning point, where $\omega/c(z) = k_n$.

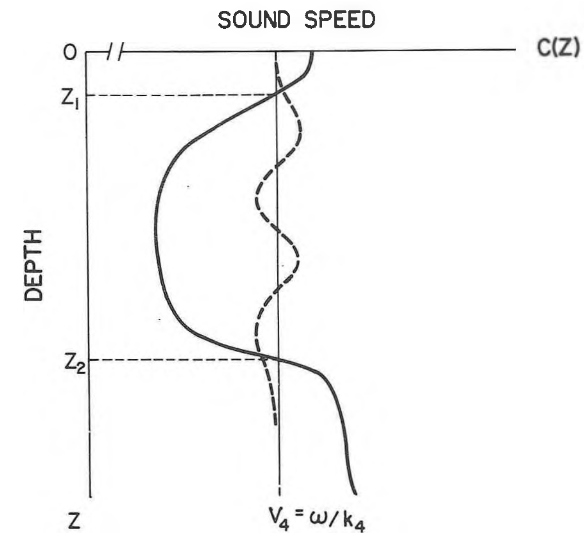

Figure 1. Normal mode solution (heavy dashed line) superimposed on a sound speed profile (heavy solid line).

Figure 1 shows a sound speed profile (heavy solid line) with a normal mode function (heavy dashed line) superimposed (with arbitrary amplitude) at the appropriate phase velocity $v_n = \omega/k_n$. Notice a number of things about the mode function:

1) At the turning points z_1 and z_2, $\omega/c(z_1) = \omega/c(z_2) = k_n$. These are the classical turning points for the equivalent ray travelling in the sound channel.

2) At the air-water interface the pressure, and hence the mode function, is zero; i.e. $u_n(0) = 0$.

3) Between 0 and z_1 the normal mode function has an increasing exponential type of behaviour.

4) Between z_2 and ∞ the solution has a decreasing exponential type of behaviour.

5) Between z_1 and z_2 there are three zero crossings since this is the fourth mode; (in general the n-th mode will have n-1 zero crossings).

Furthermore, note the sinusoidal behaviour between z_1 and z_2; note also that $u_n \simeq \sin(\pi/4)$ at the upper turning point and $u_n \simeq \sin(n-1/4)\pi$ at the lower turning point.

3 The WKB method

If the sound speed profile is changing slowly with respect to an acoustic wavelength, the WKB approximation[1] allows the solution to be written in terms of a slowly varying amplitude r(z) and a monotonically increasing phase $\phi(z)$:

$$u_n(z) = M\, r(z) \sin[\phi(z)] \tag{2}$$

where

$$r(z) = [\omega^2/c^2(z) - k_n^2]^{-1/4} \tag{3}$$

$$\phi(z) = \int_{z_1}^{z} [\omega^2/c^2(z') - k_n^2]^{1/2} dz' + \delta_1, \quad z_1 < z < z_2 \tag{4}$$

and M is a normalization constant. The integral in Eq. (4) will be referred to as the phase integral. Note several points:

1) the term in square brackets is the same term that appeared in Eq. (1).

2) near a turning point r(z) is singular.

3) $\phi(z)$ is well behaved, however, and can be used to determine the WKB eigenvalues k_n.

4) $\phi(z_1) = \delta_1$ at the upper turning point.

5) $\phi(z_2) = n\pi - \delta_2$ at the lower turning point.

Figure 2 shows how the phase δ_1 at the upper turning point depends on boundary effects. If the surface is a pressure release one, the pressure is zero, $u_n(0) = 0$ and the phase $\delta_1 = 0$; if the surface is rigid, the normal derivative of the pressure is zero, i.e. $u_n'(0) = 0$, and the phase $\delta_1 = \pi/2$. At a turning point the phase δ_1 is between 0 and $\pi/2$; $\delta_1 = \pi/4$ is the usual choice[1]. The same comments apply to the phase at the lower turning point.

In the WKB method the total phase change between the turning points is given by:

$$\int_{z_1}^{z_2} [\omega^2/c^2(z) - k_n^2]^{1/2} dz + \delta_1 + \delta_2 = n\pi. \tag{5}$$

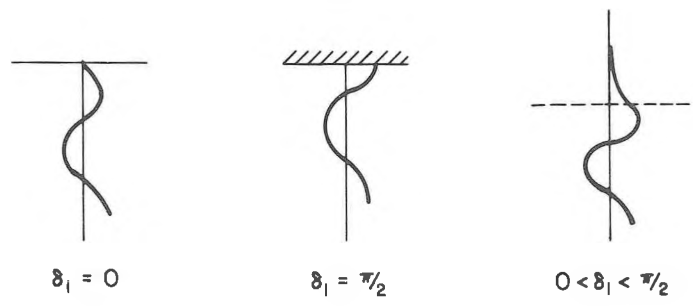

Figure 2. Effect of various boundary types on the phase δ.

where δ_1 and δ_2 are the phases at the upper and lower turning points. With the usual choice of $\delta_1 = \delta_2 = \pi/4$ the WKB eigenvalue equation becomes

$$\int_{z_1}^{z_2} [\omega^2/c^2(z) - k_n^2]^{1/2} dz = (n-1/2)\pi \tag{6}$$

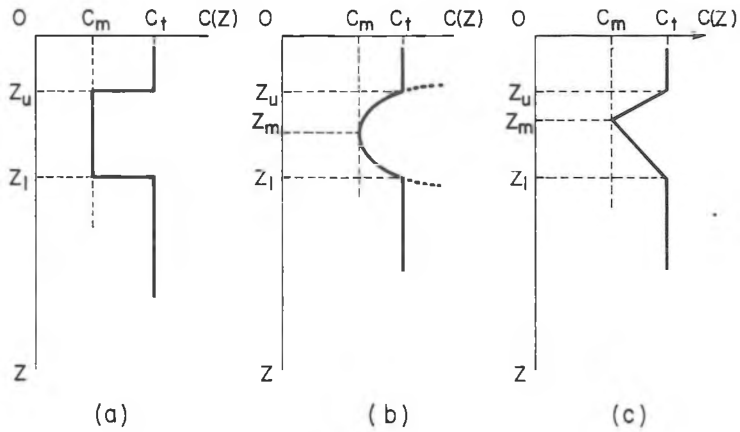

Figure 3. Three simple sound channel shapes: (a) the square isovelocity-channel profile, (b) the parabolic sound channel, and (c) the bilinear sound channel.

One wishes to solve equation (6) for k_n which appears explicitly in the integrand and implicitly in the limits z_1 and z_2. However, for certain sound speed profiles c(z) the integral can be evaluated analytically and an expression obtained for k_n. Three such sound speed profiles are shown in Figure 3 : (a) the square isovelocity-channel profile, (b) the parabolic sound channel, and (c) the bilinear sound channel. Note that it is actually c^{-2} rather than c which is parabolic or linear. Some notation is also introduced at this point: c_m = the minimum sound speed in the channel, c_t = the maximum sound speed in the channel, and $h = z_l - z_u$ = the maximum vertical extent of the sound channel.

4 The number of trapped modes

The WKB eigenvalue equation can be solved for the number of trapped modes N if the phase integral can be evaluated either analytically or numerically. Using $n \rightarrow N$, $z_1 = z_u$, $z_2 = z_l$, $k_N = \omega/c_t$ and rearranging Eq. (6) gives $h = z_l - z_u$ and $x = z/h$:

$$N = 1/2 + (2hf/c_m)[1-c_m^2/c_t^2]^{1/2} \int_0^1 \{[c_m^2/c^2(x)-c_m^2/c_t^2]/[1-c_m^2/c_t^2]\}^{1/2}\, dx \tag{7}$$

By introducing the dimensionless quantities

$$a = [1-c_m^2/c_t^2]^{1/2} \tag{8}$$

and

$$\alpha = \int_0^1 \{ [c_m^2/c^2(x)-c_m^2/c_t^2]/[1-c_m^2/c_t^2]\}^{1/2}\, dx \tag{9}$$

equation (7) can then be written as

$$N = 1/2 + (2hf/c_m)\, a\, \alpha \tag{10}$$

The quantity a is introduced strictly for notational convenience. The quantity α, however, is related to the shape of the sound speed profile, but contains none of the physical parameters such as the frequency, depth or sound speeds. Note that in the case of an isovelocity or square profile $\alpha = 1$, and Eq. (10) gives the classical formula for the number of trapped modes. Figure 4 graphically illustrates the significance of the integral in Eq. (9), where the vertical extent of the channel $z_l - z_u$ gets mapped into the range 0 to 1, and where the sound speeds between c_m and c_t get mapped into 0 to 1 and where the integrand of Eq. (9) (denoted by g(x)) is enclosed in a square box of unit size. The integral α is given by the shaded area, which can be calculated analytically or numerically or even estimated by eye.

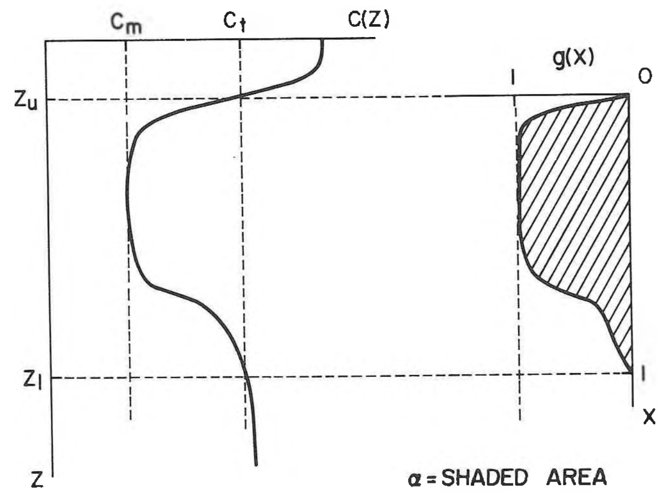

Figure 4. The sound channel and the associated function g(x); the shaded area α is defined in Eq. (9).

5 Cutoff frequencies

The cutoff frequency for the n-th mode can be obtained by rearranging Eq. (10) to give

$$f_n^{cut} = [c_m(n-1/2)] \,/\, [2ha\alpha] \tag{11}$$

6 Wave numbers

For the three sound speed profiles shown in Figure 3, the eigenvalue equation (6) can be solved for the wavenumbers k_n, giving results of the form

$$k_n^2 = \omega^2/c_m^2 - A(n-1/2)^p \tag{12}$$

where the specific values of A and p are given in Table 1. For the three profiles considered p varies between 2/3 and 2, while the corresponding values of the shape parameter α varies only between 2/3 and 1. Moreover, α enters equations (10) and (11) in a linear fashion, while p appears as an exponent in equation (12). Thus, the wave numbers are much more sensitive to the profile shape than are the trapped modes and the cutoff frequencies.

Table 1

Specific values of α, p and A for three simple profiles.

Profile Shape	α	p	A
Square	1	2	$(\pi/h)^2$
Parabolic	$\pi/4 = 0.79$	1	$4a\omega/(c_m h)$
Bilinear	2/3	2/3	$[3\pi\omega^2 a^2/(2c_m^2 h)]^{2/3}$

7 Example

Table 2 shows an example of how the formulae might be applied to a realistic ocean environment, and compares the wave numbers with those obtained from a normal mode calculation. The sound speed profile approximates a typical summer sound speed profile in 100 m of water on the Scotian Shelf: a 20 m isovelocity layer of speed 1520 m/s at the surface, a minimum sound speed of 1460 m/s at a depth of 40 m, and a speed of 1490 m/s at the bottom. The table compares the wave numbers, or phase velocities, obtained using the bilinear formula with those from a complete normal mode calculation at 200 Hz.

Table 2

Comparison of the phase velocities obtained from equation (12)

with those from a complete normal mode calculation.

Mode Number	Phase Velocities (m/s)		
	Normal mode	Equation (12)	Difference
1	1465.72	1465.46	-0.26
2	1471.34	1471.43	0.09
3	1475.99	1476.15	0.16
4	1480.13	1480.29	0.16
5	1484.94	1484.09	0.15
6	1487.48	1487.63	0.15

The WKB method gives a good approximation to the normal mode wave numbers; four or five digits accuracy as in the above example is not unreasonable. In fact for the parabolic profile the WKB and exact calculations give the identical results. Provided that this is of sufficient accuracy, the value of the analytic formula is obvious from a computational point of view.

8 Discussion

Equations (10)-(12) together with the values of α, p and A given in Table 1 summarize the results of this paper: simple analytical formulae for normal mode wave numbers, cutoff frequencies, and the number of modes trapped in a sound channel of simple shape. The factor of 1/2 appearing in Eqs. (10)-(12) can be generalized to a δ which depends on the boundaries of the sound channel as well as the type of turning point that the mode "sees". Equations (10) and (11) for the number of modes and the cutoff frequencies are useful for more general profiles provided that the integral of Eq. (9) can be estimated.

The results show the sensitivity of the modes to the shape of the sound speed profile: the number of modes depends on the shape parameter α of the sound speed profile, while the distribution of mode numbers is more sensitive to the details of the profile shape.

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