STATISTICAL MODELS IN VIBRATION ANALYSIS

by

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Introduction

Statistical models are often used in the analysis of vibration levels of complicated dynamic structures, particularly at high frequencies when the high numbers of modes involved in the vibration make both the usual modal analysis and also numerical analysis unfeasible. Results from such models often compare favourably with experimental results. There has been little effort, however, to compare the results from statistical models with exact analytic results, or to estimate analytically how well the statistical model of the structure represents the actual structure.

A particular simple coupled-beams structure is considered here (Figure 1). The vibration analysis is carried out within the framework of statistical energy analysis. In particular, results for the power flow from one beam to another are obtained. An ensemble of similar structures is constructed by varying the lengths of the beams. Exact values of the mean and standard deviation of the power flow are calculated for this ensemble. Approximate results from statistical models of the structure, including the statistical energy analysis result, are shown to compare very well with the exact results. Also, the standard deviation is found to be surprisingly small. One concludes that it is indeed justifiable when analyzing complicated dynamic structures to model not the structure itself but the simplest-to-analyze sample (in this case the symmetric case of beams of equal length) from an ensemble of structures similar to the actual one.

Statistical modelling of dynamic structures

SEA [1] is one way of estimating the dynamical response of complicated structures to high frequency excitation. In this approach the structure is treated as a number of interacting component subsystems. The energy of vibration of each sybsystem is the variable chosen to define the state of the system. Other dynamical variables such as acceleration or stress are found once the energy is known. The word statistical implies not necessarily that the excitation is random but that the parameters describing each subsystem are chosen from a known statistical distribution of similar parameters. The actual structure of interest is treated as one sample of an ensemble of similar structures.

The basic power flow and energy balance relations for a two subsystem case where only one of the sybsystems is externally excited is shown in Figure 2. P represents power, E represents the energy of the linear oscillators or modes of vibration that are used to characterise each of the subsystems and N represents the number of modes of each subsystem taking part in the energy exchange.

The power dissipated by each subsystem is related simply to the modal bandwidth Δ (which may be measured). For example $P_{1,diss} = \Delta_1 N_1 E_1$. One of the basic results of SEA is that a similar expression can be written for the power flow between the subsystems:

$$P_{12} = A_{12}(E_1 - E_2) \tag{1}$$

where A_{12} is a constant of proportionality which depends only on the dynamical parameters of the structure. With reasonable assumptions E may represent either the actual modal energies of the coupled subsystems, or the modal energies they would have if they were uncoupled but the same forces acted, although the constant A_{12}, of course, is different in each case. Results for the power flow P_{12} between the beams shown in Figure 1 are discussed below.

Considerable effort has been put into estimating the value of A_{12} for coupling between various structural components such as beams, plates and shells, and into comparing the basic SEA results such as (1) with experimental measurements. Lyon [1] lists several references on these topics. Almost all of this work is concerned with mean value estimates of the energies and power flows. Some general results for the variance of these estimates based on assumed distributions of parameters have been obtained [1,2]. Two aspects are of interest: the amount of spatial response concentration on a particular structure, and the accuracy with which the statistical model describes the actual structure. It has been shown recently [3,4] that very marked concentrations indeed are found in some cases. On the other hand, little work has been done to compare SEA results with those obtained from exact calculations for a particular structure. Remmington and Manning [5] compared mean values of power flow with an exact calculation for a few parameter values on a simple system. Smith [6] in his reworking of SEA from a wave propagation point of view has compared an SEA result with a precise statistical result. A more complete statistical treatment of the power flow in a simple structure is described below.

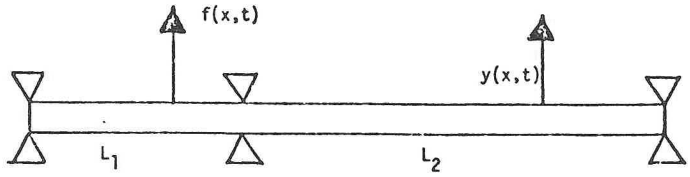

FIGURE 1

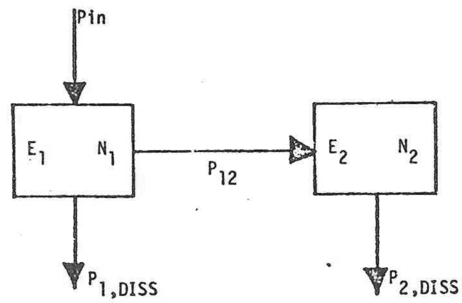

FIGURE 2

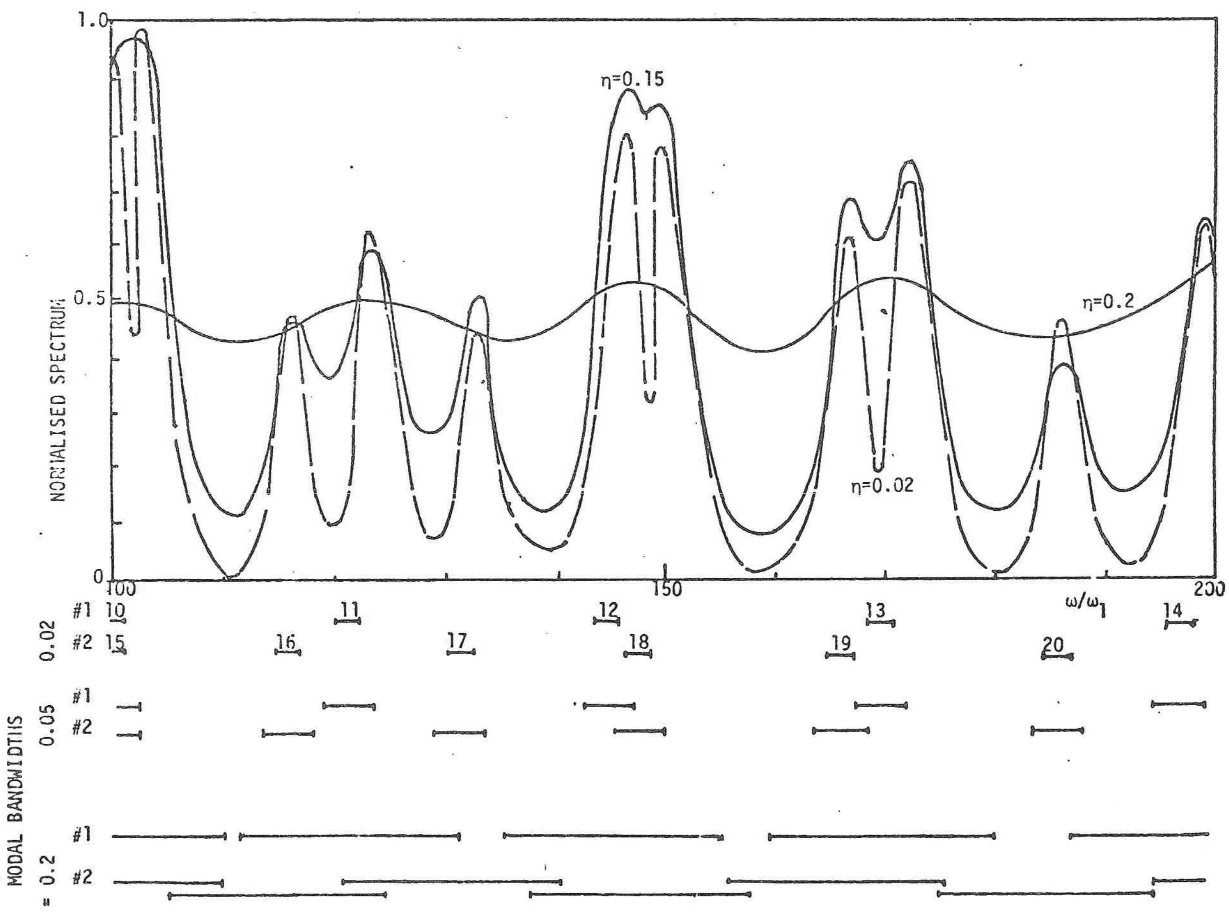

FIGURE 3

Calculations of the mean and variance of the power flow P_{12} shown in Figure 1 are discussed for a wide range of parameters. Three types of averaging are considered, ensemble, spatial and frequency, for two limiting types of random force, rain-on-the-roof and point excitation [7,8].

Power flow in a beam-beam structure

A simply supported coupled beam-beam structure is shown in Figure 1. y represents the transverse vibrational displacement response to the distributed transverse loading f(x,t). The coupling is at the central simple support. The power flow between the beams is described by the product of the bending moment and rate of change of slope at the coupling point. This structure was chosen because it is sufficiently complicated to exhibit many of the multi-modal interactions inherent in the SEA assumptions yet sufficiently simple that an exact solution can easily be obtained, at least within the limits of the Bernoulli-Euler bending theory.

Two limiting types of random forces are considered here: rain-on-the-roof where f(x,t) is white in both space and time, and a point force at the point x = z so that $f(x,t) = f_o(t)\ \delta(x-z)$ where $f_o(t)$ represents a white noise force. In both cases only beam 1 is externally excited.

The problem can be solved either in terms of the simple support-simple support eigenfunctions of the beams or by using a closed form Green's function. For rain-on-the-roof excitation the modal forces are uncorrelated. This fact was used in [7] to obtain an eigenfunction expansion for the power flow which was subsequently summed. For point excitation a closed-form Green's function is obviously more appropriate. A Green's function for the beam-beam structure was obtained in [8] by superposing results for the point force at x=z and an unknown force at the coupling point, and evaluating the unknown force from the requirement that the displacement be zero at the coupling poing.

Expressions for the spectrum of the power flow, or more strictly for the frequency decomposition of that part of the power flow at the coupling point that can be dissipated in beam 2 can be written in non-dimensional variables in the form

$$\frac{\Pi_{12}(\omega)}{E_1-E_2} = \mathrm{fn}\ \left(\frac{\omega}{\omega_1}, \frac{L_2}{L_1}, \frac{z}{L_1}, \frac{\Delta}{\omega_1}\right) \tag{2}$$

Here ω_1 represents the fundamental resonance frequency of beam 1 and η the loss factor which is assumed the same for both beams. The z/L_1 dependence occurs of course only for the point excitation case. Δ/ω_1 is a damping loss factor.

The functional form in (2) is sufficiently complicated that general results concerning changes in structural parameters that may be required in a design process are not easy to determine. Various statistical averages of the spectrum can be considered instead for given values of the loss factor. Frequency averages over ω/ω_1 are typically over one-third octave or octave bandwidths. Since in many if not most cases of practical interest energy exchange between subsystems is by resonant modes the important parameter in frequency averages is not the averaging bandwidth itself but the number of modes that have resonance frequencies within the band. Averages over L_2/L_1 can be thought of as ensemble averages. The ensemble of similar systems considered here has a uniform distribution of L_2/L_1 between the values one and two. Spatial averages are taken over the variable z/L_1. It is assumed here that z/L_1 is uniformly distributed between the values zero and one.

Numerical Results

Figures 3 and 4 are for the case of rain-on-the-roof excitation. Figure 5 includes results for both types of excitation considered here. Typical values of the normalised spectrum of power flow are shown in Figure 3 for $L_2=1.4L_1$ for three values of the loss factor η. The corresponding value of the modal overlap ratio M, the ratio of modal bandwidth to average spacing between resonance frequencies is also shown. The bars underneath the graph show the modal bandwidths for each mode of the uncoupled beams.

The modal nature of the response is seen clearly, and it is evident that at low values of the damping the power flow is predominantly by resonant mode interaction. Proximate modes whose bandwidths overlap give high values of the power flow spectrum. As the modal bandwidth increases the power flow in the octave band shown in Figure 3 increases, at least in the range $M < 1$. Figure 4 shows that the relation is in fact linear. Considerable smoothing of the spectrum occurs as M increases. For the case $M = 1.2$ shown the spectrum varies only slightly about a mean value of almost 0.5. The SEA result due to Lotz [9] using Lyon's [1] wave propagation approximation gives a value of 0.5 for this problem.

The graphs in Figure 3 emphasise the complicated nature of the function in equation (2). Meaningful results concerning structural parameter dependence can only be obtained in general terms if suitable average values of the power flow are considered.

It is shown in references [7] and [8] that some smoothing of the spectrum is obtained by ensemble averaging over an "octave" of structures $1 \leq L_2/L_1 \leq 2$. Marked peaks in the ensemble spectrum still occur near the resonance frequencies of beam 1, and it is still not easy to determine the effect of various structural parameters on the results. On the other hand, if frequency averages are taken, the rain-on-the-roof octave band spectrum is rather insensitive to changes in L_2/L_1.

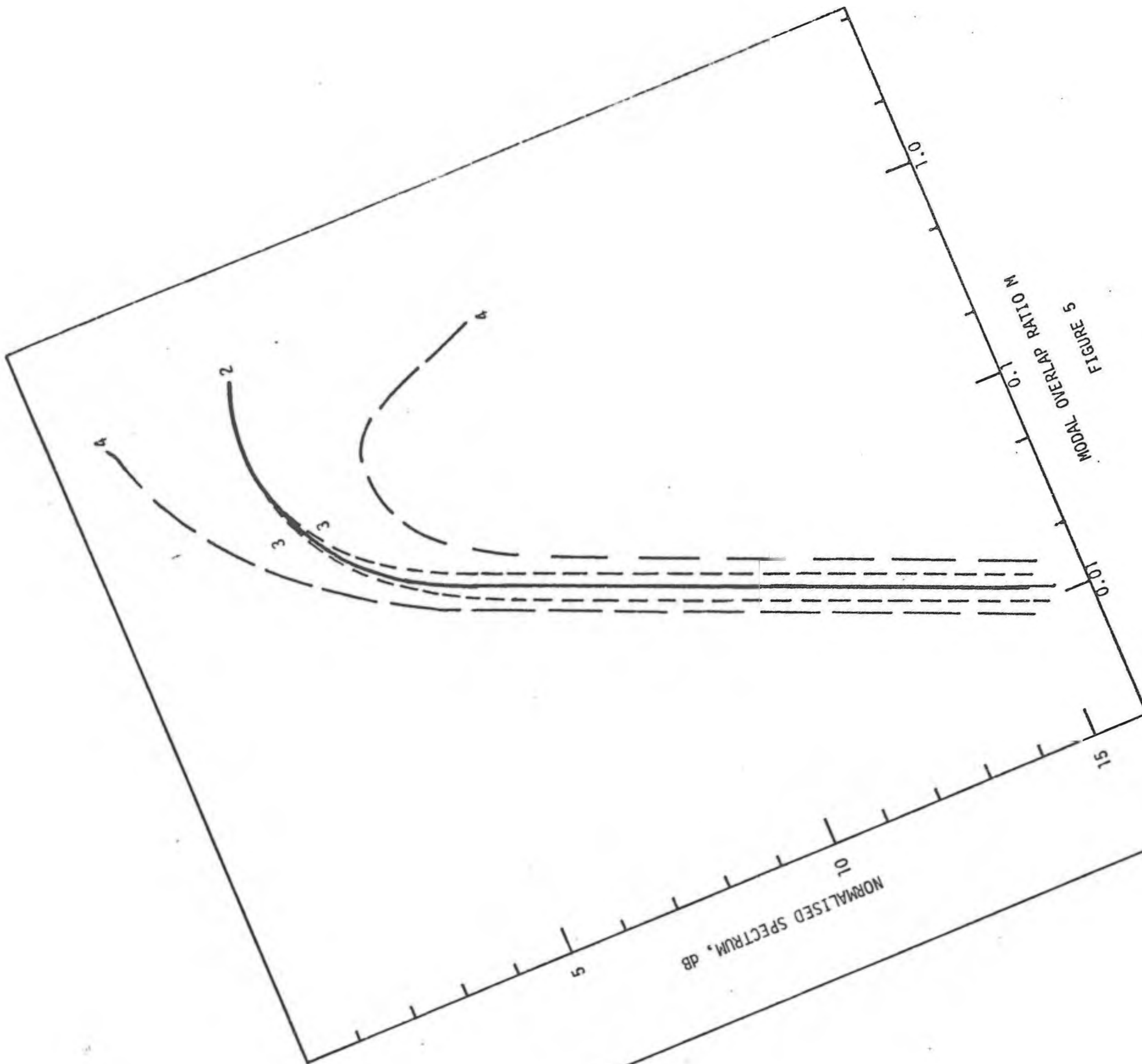

FIGURE 5

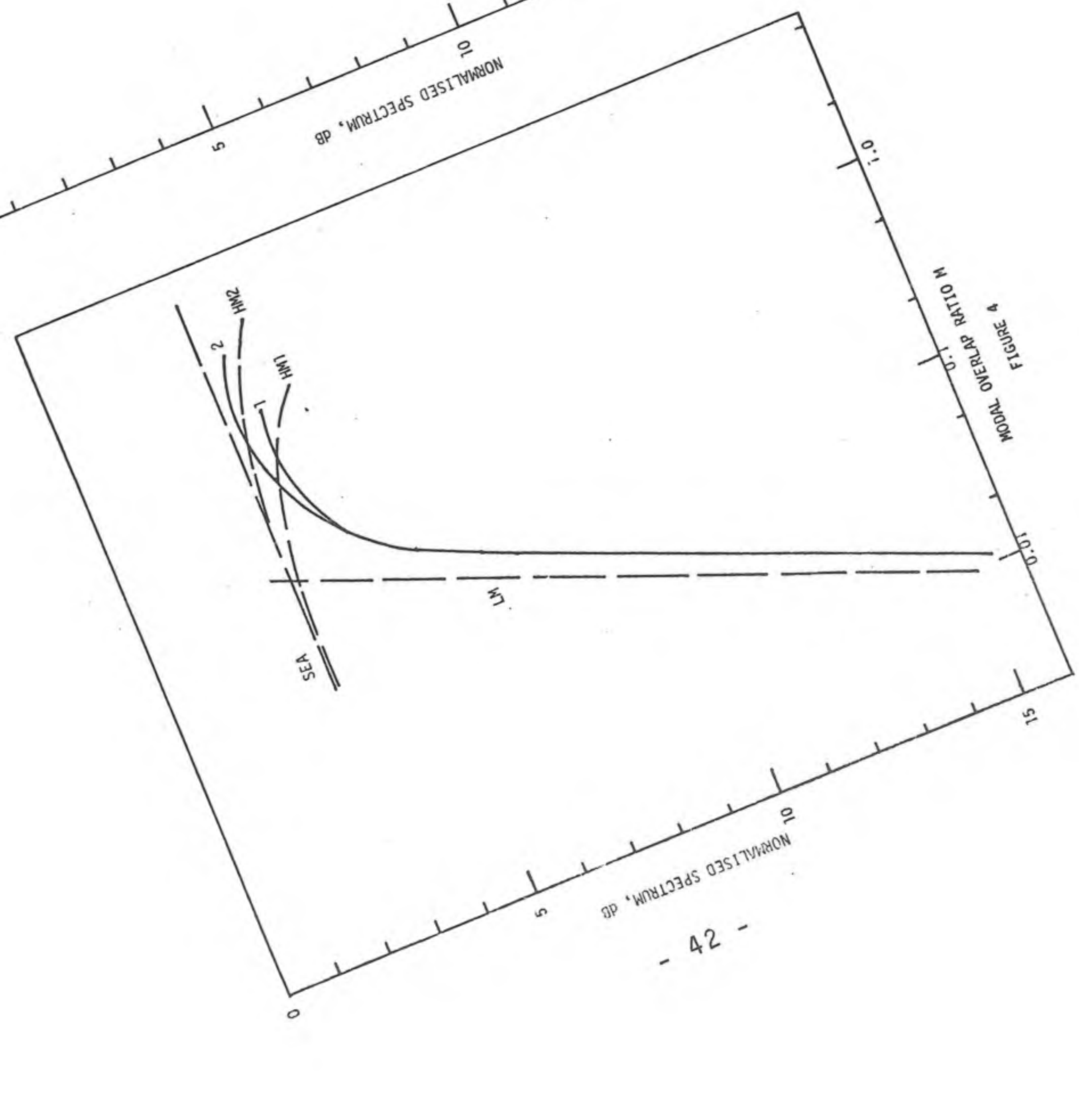

FIGURE 4

Octave band results for rain-on-the-roof excitation are shown in Figures 4 and 5. Frequency averaged values of the spectrum of power flow are plotted against the modal overlap ratio for two octave bands. Curves labelled 1 and 2 in Figure 4 are for $10 < \omega/\omega_1 < 20$ and $100 < \omega/\omega_1 < 200$, respectively. These are the exact octave band mean values for the ensemble $1 \leq L_2/L_1 \leq 2$. For $M \ll 1$ the curves are the same. In the region $M = 1$ and for $M > 1$ there are small differences. Figure 5 shows the mean ± standard deviation (curve 3) for curve 2, the deviation being over the ensemble $1 \leq L_2/L_1 \leq 2$. The deviation is surprisingly small. For curve 2 there are 4.8 resonant modes in the octave band. For curve 1 there are only 1.3 resonant modes. The deviation (not shown) even in this case is still surprisingly small.

Several approximate results are also shown in Figure 4. The SEA result is due to Lotz [9] and as expected shows good agreement with the exact results when the modal overlap ratio is high. The curves LM and HM are approximate results for the case $L_2 = L_1$. The symmetry in this case makes the calculation simpler. The curve LM is obtained by considering only the resonant mode contribution from the perfectly overlapping modes of the two systems [7]. Curves HM1 and HM2 are obtained by assuming that because of the smoothing caused by high modal overlap, summations over modes can be replaced by integrals [7].

Figure 5 shows standard deviations of the octave band power flow for the case $100 < \omega/\omega_1 < 200$. Curve 2 is the same as in Figure 4 for the rain-on-the-roof excitation. Curves 3 are the mean ± standard deviation for this case. Curves 4 are the mean ± standard deviation for the octave band power flow in the point force case when $L_2 = L_1\sqrt{2}$, the deviation being taken over the spatial average $0 \leq z \leq L_1$. The mean value for this case is so close to curve 2 that it has not been shown. For rain-on-the-roof excitation the smoothing caused by high modal overlap makes the standard deviation very small. High modal overlap here implies high damping. For $M \geq 1$, the damping is so high that there is appreciable decay of waves along beam 1 emanating from the point of excitation. For small z there is very little energy incident on the coupling point so the power-flow through the coupling point is small. On the other hand, for $z \simeq L_1$ there is considerable energy incident on the coupling point. The deviation over the spatial average thus increases as M increases. This decay effect for the point excitation case is examined in more detail in reference [8].

Conclusion

Figures 4 and 5 add considerable credence to the use of statistical models in vibration analysis. Excellent agreement is obtained if octave band values are considered adequate. As expected the SEA wave transmission result [9] which assumes in essence that the beams are infinitely long shows good agreement with exact values when the modal overlap ratio is high. Because the standard deviation is so small, a statistical model using equal beams and uniformly spaced perfectly overlapping modes shows good agreement for all values of M, although different approximations must be used for M<1 and M>1. Again provided octave band values are used results for the spatially averaged point force case agree well with those for the rain-on-the-roof case.

Further work is in progress at UNB extending the results to more complicated structures and examining spatial concentration of vibration levels along the beams. It is expected that this work will confirm the conclusions outlined above. One may conclude it indeed seems justifiable that when analyzing the vibration of a complicated structure one may use results obtained from the simplest to analyze (for example, a symmetric case) sample from an ensemble of similar structures.

Acknowledgement

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